Laplace–Iman Transform and its Properties with Applications to Integral and Partial Differential Equations.

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Abstract: Laplace Iman transform (LIT) as a double integral transform of a function $f(x,t)$ of two variables was presented to solve some integral and partial differential equations. Main properties and theorems were proved. The convolution of two function $f(x,t)$ and $g(x,t)$ and the convolution theorem were discussed. The integral and partial differential equations were turned to algebraic ones by using ((LIT) and its properties. The results showed that the Laplace-Iman transform was more efficient and useful to handle such these kinds of equations.

Keywords: Laplace–Iman transform, Laplace transform, Iman transform, Convolution, Integral and partial differential equations.

1. Introduction :

Pierre – Simon Laplace (1782) introduced the idea of its transform that became one of the most famous transforms in mathematics, physics and engineering sciences. The Laplace transform was used to find out the solution of linear differential, difference, and integral equations ,On ther hand, Iman Ahmed Almardy modified Sumudu transform ,and gave a new technique that used to solve linear and nonlinear differential equations ,integral equations], and other applications .Whereas using single Laplace and Iman transforms to solve equations with unkown function of two variables were hard and useless sometimes so the matheticians as L.Debnath used the double Laplace transform to solve functional, double integral equations and partial differential equations .While I.A.ALmardy solved telegraph partial differential equation by using double Iman transform .In this paper , another double transform which called Laplace-Iman transform of function $f(x,t)$ was studied with properties and theorems to find out the solution of some integral and partial differential equations. Consequently, the aim of this work is to develop a method to solve double integral and partial differential equations easily by turning these kind of equations to algebraic ones.

2. Research problem:

Partial differential equations and double integral equations with convolution type are used to describe many problems in the field of engineering and most applied science . The solving of these equations by Using single transforms were more difficult than using the double transforms.

The Laplace-Iman transform as a double transform and its properties were discussed to solve these kind of equations easily.

3. Materials and methods of research:

The inductive thinking was used to get a double integral transform. The single Laplace and Iman transforms were combined in a double integral transform which called Laplace-Iman transform , so most of properties for the two single transforms were generalized and most results related to the gamma function.

3.1. Basic concepts:

3.1.1. Laplace transform :

Let $f(x)$ is a piecewise continuous function on interval $[0,\infty[$ and of exponential order so it satisfy :

$$|f(x)| < Me^{ax}\,;\; a > 0$$

Then the Laplace transform of this function is defined by :

$$L(s)=L(f(x)) =\int_0^\infty e^{-sx}f(x)dx \qquad (1)$$

The inverse Laplace transform is :

$$f(x)=L^{-1}(L(s)) = \int_0^\infty e^{sx}L(s)ds \qquad (2)$$

3.1.2. Iman transform :

If $f(t)$ is a function from the set A where:

$$A=\{f(t): \exists M,k_1,k_2>0,|f(t)|<Me^{\frac{|t|}{k_j}};\ t\in(-1)^j\times[0,\infty[\ \}$$

Then the Iman transform of this function is:

$$\mathrm{I}[f(t)]=\frac{1}{p^2}\int_0^{\infty} f(t)e^{-tp^2}dt \qquad (3)$$

And the relation between Laplace and Iman transform is :

$$L(s)=\frac{1}{p^2}\ I(p^2) \qquad (4)$$

So the inverse Iman transform has the form :

$$f(t)=I^{-1}\big(I(p)\big)=\int_0^{\infty}\quad e^{st}\frac{1}{p^2}\ I(p^2)\ dp \qquad (5)$$

3.1.3. Gamma function:

$\Gamma(a)$ is the gamma function which defined by the form

$$\Gamma(a)=\int_0^{\infty} e^{-x}\,x^{a-1}dx \qquad (6)$$

The gamma function satisfies many properties like :

$$\Gamma(a+1)=a\Gamma(a) \qquad (7)$$

If $a\in N$ then,

$$\Gamma(\mathrm{a})=(\mathrm{a}+1)! \qquad (8)$$

4. Discussion and results:

4.1. Definition of the Laplace-Iman transform (LIT) :

The Laplace-Iman transform of function $f(x,t)$ of two variable x and t defined in the first quadrant of the $x-t$ plane is defined by the double integral in the form:

$$\bar{\bar{f}}\ (s,p)=L_x\mathrm{I}_t(f(x,t))=L\mathrm{I}(f(x,t))=\frac{1}{p^2}\int_0^{\infty}\int_0^{\infty} e^{-sx-tp^2}\ f(x,t)dxdt \qquad (9)$$

Evidently, $L\mathrm{I}T$ is linear integral transformation as shown below :

$$L\mathrm{I}(\alpha f(x,t)+\beta g(x,t))=\frac{1}{p^2}\int_0^{\infty}\int_0^{\infty} e^{-sx-tp^2}\ (\alpha\mathrm{f}(\mathrm{x},\mathrm{t})+\beta\mathrm{g}(\mathrm{x},\mathrm{t}))dxdt$$

$= \frac{1}{p^2}\int_0^\infty \int_0^\infty e^{-sx-tp^2}\, \alpha f(x,t)dxdt + \frac{1}{p^2}\int_0^\infty \int_0^\infty e^{-sx-tp^2}\, \beta g(x,t)dxdt$

$= \frac{\alpha}{p^2}\int_0^\infty \int_0^\infty e^{-sx-tp^2}\, f(x,t)dxdt + \frac{\beta}{p^2}\int_0^\infty \int_0^\infty e^{-sx-tp^2}\, g(x,t)dxdt$

$=\alpha LI(f(x,t))+\beta LI(g(x,t))$ (10)

Where α and β are constants.

By using the Bromwich inversion formula , The inverse Laplace-Iman transform is defined by the complex integral formula :

$$\frac{1}{(2\pi i)^2}\int_{\alpha-i\infty}^{\alpha+i\infty}\int_{\beta-i\infty}^{\beta+i\infty} \frac{1}{p^2} e^{sx+tp^2} f(s,p^2)dsdp \qquad (11)$$

4.2. Laplace – Iman transform of basic functions :

(a)

$f(x,t) = x^a t^b$ for $x > 0$ and t>0, then

$$f''(s,p) = LI(x^a t^b) = \frac{1}{p^2}\int_0^\infty \int_0^\infty e^{-sx-tp^2} x^a t^b \, dxdt$$

$$= \int_0^\infty x^a e^{-sx} dx \int_0^\infty \frac{1}{p^2} e^{-tp^2} t^b \, dt$$

$$= \frac{\Gamma(a+1)}{s^{a+1}} p^{-2b-4} \Gamma(b+1) \qquad (12)$$

In particular,

$$LI(1) = \frac{1}{s\,p^2} \qquad (13)$$

where a>−1 and b>−1 are real numbers
Consequently if a and b are natural numbers in (12) we obtain

$$LI(x^a t^b\ ) = \frac{a!\,b!}{s^{a+1}} p^{-2b-4} \qquad (14)$$

(b)

$$LI\big(J_0(a\sqrt{xt})\big) = \frac{1}{p^2}\int_0^\infty \int_0^\infty e^{-sx-tp^2}\, J_0(a\sqrt{xt})dxdt$$

$$= \int_0^\infty e^{-sx} J_0(a\sqrt{xt})dx \int_0^\infty \ \frac{1}{p^2} e^{-tp^2} t^b \ dt$$

$$= \frac{1}{s\,p^2}\int_0^\infty \ e^{-tp^2} e^{-\frac{ta^2}{4s}}\, dt = \frac{1}{s} I(e^{-\frac{ta^2}{4s}}) \qquad (15)$$

(C)

$$LI[H(x-x_0,t-t_0)] = \frac{1}{p^2}\int_0^\infty \int_0^\infty e^{-sx-tp^2}\, H(x-x_0,t-t_0)dxdt$$

$$= \int_{x_0}^\infty e^{-sx}dx \int_{t_0}^\infty \ \frac{1}{p^2} e^{-tp^2} t^b \ dt$$

$$= \frac{1}{p^2} e^{-t_0 p^2} e^{-\frac{sx_0}{s}} \qquad (16)$$

where $H(x-x_0,t-t_0)$ is Heaviside function which has the formula :

$$H(x-x_0,t-t_0) = \begin{cases} 1\ ;\ x \geq x_0 \text{ and } t \geq t_0 \\ 0\ ;\ x < x_0 \text{ and } t < t_0 \end{cases} \qquad (17)$$

A table of Laplace-Iman transform can be constructed from the standard tables of Laplace and Iman transforms by using the definition (9) or by evaluating double integrals. The above results can be used to solve integral and partial differential equations.

4.2.1. Corollary: Let $f(x,t)=g(x)h(t)$, then

$$f''(s,p)=LI[f(x,t)]=L[g(x)]I[h(t)] \tag{18}$$

Proof: $f''(s,p)=LI[f(x,t)]=$

$$\frac{1}{p^2}\int_0^\infty\int_0^\infty e^{-sx-tp^2}\,\mathrm{g(x)h(t)}\,dxdt$$

$$=\int_0^\infty e^{-sx}\,g(x)dx\int_0^\infty \frac{1}{p^2}e^{-tp^2}h(t)\,dt$$

$$=L[g(x)]\,I[h(t)] \tag{19}$$

Derivative property :

If $f''(s,p)=LI(f(x,t))$ then,

(a) $LI\left[\frac{\partial f(x,t)}{\partial x}\right]=Sf''(\mathrm{s,p})-If(0,t)$

Proof:

$$LI\left[\frac{\partial f(x,t)}{\partial x}\right]=\frac{1}{p^2}\int_0^\infty\int_0^\infty e^{-sx-tp^2}\frac{\partial f(x,t)}{\partial x}\,dxdt$$

$$=\frac{1}{p^2}\int_0^\infty e^{-sx}\,\frac{\partial f(x,t)}{\partial x}dx\int_0^\infty e^{-tp^2}\,dt$$

Suppose $u=e^{-sx}$,$dv=\frac{\partial f(x,t)}{\partial x}dx$ then,

$$= Sf^{''}(\mathrm{s},\mathrm{p}) - If(0,t)$$

(b) $LI\left[\frac{\partial f(x,t)}{\partial t}\right] = p^2 f^{''}(\mathrm{s},\mathrm{p}) - \frac{1}{p^2} L\, f(x,0)$

Proof:

$$LI\left[\frac{\partial f(x,t)}{\partial x}\right] = \frac{1}{p^2}\int_0^\infty \int_0^\infty e^{-sx-tp^2}\frac{\partial f(x,t)}{\partial t}\, dxdt$$

$$= \frac{1}{p^2}\int_0^\infty e^{-sx}\, dx \int_0^\infty \quad e^{-tp^2}\frac{\partial f(x,t)}{\partial t}\, dt$$

Suppose $u{=}e^{-tp^2}$,$dv{=}\frac{\partial f(x,t)}{\partial t} dt$ then,

$$= p^2 f^{''}(\mathrm{s},\mathrm{p}) - \frac{1}{p^2} L\, f(x,0)$$

Similarly,

(C) $\mathrm{LI}\left[\frac{\partial^2 f(x,t)}{\partial x \partial t}\right] = sp^2 f''(\mathrm{s},\mathrm{p}) - \frac{s}{p^2} L(\, f(x,0)) - I(f_t(0,t))$

(d) $\mathrm{LI}\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right] = s^2 f''(\mathrm{s},\mathrm{p}) - sI(\, f(0,t)) - I(f_x(0,t))$

(e) $\mathrm{LI}\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = p^2 f''(\mathrm{s},\mathrm{p}) - L(\, f(x,0)) - \frac{1}{p^2} L(f_t(x,0))$

Convolution and convolution theorem of the Laplace-Iman transform:

The convolution of $f(x,t)$ and $g(x,t)$ is denoted by $(f{*}{*}g)(x,t)$ and defined by :

$$(f{*}{*}g)(x,t) = \int_0^x \int_0^t \mathrm{f}(\mathrm{x}-\mathrm{u},\mathrm{t}-\mathrm{v})\, \mathrm{g}(\mathrm{u},\mathrm{v})\mathrm{dudv}$$

The convolution is commutative, that is:

$$(f{*}{*}g)(x,t) = (g{*}{*}f)(x,t)$$

This follows from the definition it can easily be verified that the following properties of convolution are correct :

1. $[f**(g**h)](x,t)=[(f**g)**h](x,t)$ (Associative)
2. $[f**(ag+bh)](x,t)=a(f**g)(x,t)+b(f**h)(x,t)$ (Distributive).
3. $(f**\delta)(x,t)=(\delta**f)(x,t)=f(x,t)$ (Identity).

Where $\delta(x,t)$ is the Dirac delta function of x and t.

Conclusion:

Two different single transforms were combined together to present a new double transform (LIT), so main properties and theorems were generalized with proofs to solve some integral and patial differential equations. We concluded that (LIT) was a powerful technique to deal with these kind of equations. Another applications on integro-partial differential and nonlinear partial differential equations will be discussed in subsequent paper.

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Pages : 1487-1494

Journal Name: Frontiers in Health Informatics

Solution of Linear and Nonlinear Partial Differential Equations Using Mixture of IMAN Transform and the Projected Differential Transform Method

URL : https://healthinformaticsjournal.com/index.php/IJMI/article/view/2026

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Table of Functions and their Iman Transform

$f(t)$	$I[f(t)] = F(p)$
1	$\frac{1}{p^4}$
t	$\frac{1}{p^6}$
t^2	$\frac{2!}{p^8}$
$t^n \quad n \in N$	$\frac{n!}{p^{2n+4}}$
e^{at}	$\frac{1}{p^2(p^2 - a)}$
$\sin(at)$	$\frac{a}{p^2(p^4 + a^2)}$
$\cos(at)$	$\frac{1}{p^4 + a^2}$