Seventh order second derivative method with optimized hybrid points for solving first-order initial value problems of ODEs

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Abstract

A seventh order second derivative method with optimized hybrid points is proposed for the solution of first-order ordinary differential equations. The techniques of interpolation and collocation are employed for the construction of the method using a three-parameter representation of the hybrid points. By optimizing the local truncation error of the main method, the hybrid points are obtained and then used to derive second derivative method. The discrete schemes are produced as by-products of the continuous scheme and used to simultaneously solve initial value problems (IVPs) in block mode. The resulting schemes are self-starting, consistent, zero-stable, and A-Stable. The accuracy of the method was established using four test problems. The numerical results revealed that the new method performed better than existing methods in the cited literature.

Keywords: Initial Value Problems (IVPs), Local truncation error (LTE), Ordinary Differential Equations (ODEs), Parameter approximations, Second derivative

1 Introduction

Differential equations are produced through the mathematical modeling of physical processes in the scientific and engineering fields, particularly in epidemiological systems with numerous interactions among various compartments. The analytical solutions to the majority of differential equations are typically difficult to find. This demanded the use of numerical techniques to provide an approximate solution. Traditionally, the numerical approximations of the exact solutions to ODEs are obtained using Runge-Kutta and linear multi-step approaches. However, the constant research in this field focuses on creating new techniques that are both efficient and simple in structure, while also possessing strong stability features.

The purpose of this article is to create and evaluate a new and effective numerical integration method for solving initial value problems of the form

$$v' = f(t, v), v(t_0) = v_0 \tag{1}$$

where t is a variable in the interval $[t_0, T]$, $v : [t_0, T] \rightarrow R$, and $f : [t_0, T] \times R \rightarrow R$. Firstly, we assume that equation (1) fulfills the requirements outlined in the Existence and Uniqueness Theorem for initial-value problems [24]. To numerically solve equation (1), the interval of integration $[t_0, T]$ is discretized by dividing it into smaller intervals. Each interval is represented by $t_n = t_0 + nh$, where n is an integer starting from 0 and h represents the difference between t_{n+1} and t_n. The step-lengths, nh, can be chosen as either a constant or a variable within the desired interval, depending on how the integrator is implemented. v_n represents the approximate numerical value of the theoretical solution at t_n, which is represented as $v(t_n)$. In addition to the Runge-Kutta and linear multistep methods, there are additional well-recognized categories of methods for numerical integration, such as block methods, hybrid methods, exponentially fitted methods, and trigonometrically fitted methods. To obtain a comprehensive analysis of various categories of techniques, individuals can refer to the publications authored by [10] or by [16] (as well as the cited sources within). One might resort to specific sources that provide solutions for various sorts of differential equations, such as the ones mentioned in [1-9,11-15,17-24].

The numerical method employed in this research is a hybrid approach that combines both block and hybrid characteristics. Dahlquist's first barrier imposes limitations on the number of steps and the order of stable linear multi-step techniques [22]. Based on this criterion, a linear multi-step technique that is stable at zero would have an order of p, where $p \leq k + 1$ if k is an odd number, and $p \leq k + 2$ if k is an even number. To overcome this obstacle, numerous writers have suggested hybrid approaches that use information from solution locations that are not in the current step. By utilizing data from off-step sites, it is possible to circumvent Dahlquist's initial barrier. These techniques are alternatively referred to as modified linear multi-step algorithms [1]. In contrast, block approaches generate solution information at multiple sites concurrently.

This study presents novel approach called the Second Derivative Method with Optimized Hybrid Points (SDMOHP).

2. Development of the method

Suppose that the exact solution $v(t)$ of equation (1) is approximated by the polynomial $p(t)$ given by

$$p(t) = \sum_{j=0}^{m} w_j t^j. \tag{2}$$

where $w_j \in R$ are real undetermined coefficients. $m = (C + I) - 1$, I is the number of interpolation and C is the number of collocation points. Differentiating (2) yield the first derivative as

$$p'(t) = \sum_{j=0}^{m} j w_j t^{j-1}. \tag{3}$$

Differentiating (3) gives the second derivative as

$$p''(t) = \sum_{j=0}^{m} j(j-1) w_j t^{j-2}. \tag{4}$$

Interpolating equation (2) at $t_{n+j}, j = 0$ and collocating equation (3) at $t_{n+j}, j = 0, u_1, u_2, u_3, 1$, where u_1, u_2, u_3 are the hybrid points such that $0 < u_1 < u_2 < u_3 < 1$.This yields a system of linear equations given as

$$\begin{pmatrix} 1 & t_n & t_n^2 & t_n^3 & t_n^4 & t_n^5 \\ 0 & 1 & 2t_n & 3t_n^2 & 4t_n^3 & 5t_n^4 \\ 0 & 1 & 2t_{n+u_1} & 3t_{n+u_1}^2 & 4t_{n+u_1}^3 & 5t_{n+u_1}^4 \\ 0 & 1 & 2t_{n+u_2} & 3t_{n+u_2}^2 & 4t_{n+u_2}^3 & 5t_{n+u_2}^4 \\ 0 & 1 & 2t_{n+u_3} & 3t_{n+u_3}^2 & 4t_{n+u_3}^3 & 5t_{n+u_3}^4 \\ 0 & 1 & 2t_{n+1} & 3t_{n+1}^2 & 4t_{n+1}^3 & 5t_{n+1}^4 \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{pmatrix} = \begin{pmatrix} v_n \\ f_n \\ f_{n+u_1} \\ f_{n+u_2} \\ f_{n+u_3} \\ f_{n+1} \end{pmatrix}. \qquad (5)$$

Solving equation (5), the coefficients w_j 's, $j = 0,1,...,5$ are obtained and substituted into equation (3) to get the implicit scheme of the form:

$$v(t) = \alpha_0(t)v_n + h(\beta_0(t)f_n + \beta_{u_1}(t)f_{n+u_1} + \beta_{u_2}(t)f_{n+u_2} + \beta_{u_3}(t)f_{n+u_3} + \beta_1(t)f_{n+1}. \qquad (6)$$

where, $\alpha_0(t)$, and $\beta_j(t), j = 0, u_1, u_2, u_3, 1$ are continuous coefficients.

Evaluating equation (5) at the points $t = t_{n+u_1}, t_{n+u_2}, t_{n+u_3}, t_{n+1}$, yield the following

$$\begin{aligned} v_{n+u_1} &= v_n + \frac{hu_1(-3u_1^3 + 30u_2u_3 + 5u_1^2)(1 + u_2 + u_3) - 10u_1(u_2 + u_3 + u_2u_3))f_n}{60u_2u_3} \\ &+ \frac{hu_1^3(3u_1^2 + 10u_2u_3 - 5u_1(u_2 + u_3))f_{n+1}}{60(-1 + u_1)(-1 + u_2)(-1 + u_3)} \\ &+ \frac{hu_1\left(12u_1^3 - 30u_2u_3 + 5u_1^2(1 + u_2 + u_3) + 20u_1(u_2 + u_3 + u_2u_3)\right)f_{n+u_1}\,)}{60(-1 + u_1)(u_1 - u_2)(u_1 - u_3)} \\ &+ \frac{hu_1^3(3u_1^2 + 10u_3 - 5u_1(1 + u_3))f_{n+u_2}}{60(u_1 - u_2)(-1 + u_2)u_2(u_2 + u_3)} + \frac{hu_1^3\left(3u_1^2 + 10u_2 - 5u_1(1 + u_2)\right)f_{n+u_3}}{60(u_1 - u_3)(-1 + u_3)u_3(-u_2 + u_3)}, \end{aligned} \qquad (7)$$

$$\begin{aligned} v_{n+u_2} &= v_n + \frac{hu_2(5u_1\left(u_2^2 + 6u_3 - 2u_2(1 + u_3)\right) + u_2(-3u_2^2 - 10u_3 + 5u_2(1 + u_3))f_n}{60u_1u_3} \\ &+ \frac{hu_2^3\left(u_2(3u_2 - 5u_3) - 5u_1(u_2 - 2u_3)\right)f_{n+1}}{60(-1 + u_1)(-1 + u_2)(-1 + u_3)} \\ &- \frac{hu_2^3\left(3u_2^2 + 10u_3 - 5u_2(1 + u_3)\right)f_{n+u_1}}{60(-1 + u_1)u_1(u_1 - u_2)(u_1 - u_3)} \\ &+ \frac{hu_2(5u_1\left(3u_2^2 + 6u_3 - 4u_2(1 + u_3)\right) + s(-12u_2^2 - 20u_3 + 15u_2(1 + u_3))f_{n+u_2}}{60(u_1 - u_1)(-1 + u_2)(u_2 - u_3)} \\ &- \frac{hu_2^3(5u_1(-2 + u_2) + (5 - 3u_2)u_2)f_{n+u_3}}{60(u_1 - u_3)(-1 + u_3)(-u_2 + u_3)}, \end{aligned} \qquad (8)$$

$$
\begin{aligned}
&v_{n+u_3}\\
&= v_n\\
&+ \frac{hu_3\big(u_3(5u_2(-2+u_3)+(5-3u_3)u_3)+5u_1(-2u_2(-3+u_3)+(-2+u_3)u_3)\big)f_n}{60u_1u_2}\\
&+ \frac{hu_3^3(10u_1u_2-5u_1u_3-5u_2u_3+3u_3^2)f_{n+1}}{60(-1+u_1)(-1+u_2)(-1+u_3)} + \frac{hu_3^3(5u_2(-2+u_3)+(5-3u_3)u_3)f_{n+u_1}}{60(-1+u_1)u_1(u_1-u_2)(u_1-u_3)} \qquad (9)\\
&- \frac{hu_3^3(5u_1(-2+u_3)+(5-3u_3)u_3)f_{n+u_2}}{60(u_1-u_2)(-1+u_2)(u_2-u_3)}\\
&+ \frac{hu_3(u_3(3(5-4u_3)u_3+5u_2(-4+3u_3))+5u_1\big(u_2(6-4u_3)+u_3(-4+3u_3)\big))f_{n+u_3}}{60(u_1-u_3)(-1+u_3)(-u_2+u_3)},
\end{aligned}
$$

$$
\begin{aligned}
&v_{n+1}\\
&= v_n + \frac{h\big(-3+u_2(5-10u_3)+5u_2+5u_1(1-2u_3+u_2(-2+6u_3))\big)f_n}{60u_1u_2u_3}\\
&+ \frac{h(= 12+15u_2+15u_3-20u_2u_3+5u_1(3-4u_3+u_2(-4+6u_3)))f_{n+1}}{60(-1+u_1)(-1+u_2)(-1+u_3)} \qquad (10)\\
&+ \frac{h(3-5u_3+5u_2(-1+2u_2))f_{n+u_1}}{60(-1+u_1)u_1(u_1-u_2)(u_1-u_3)} + \frac{h(3-5u_3+5u_1(-1+2u_2))f_{n+u_2}}{60(u_1-u_2)(-1+u_2)(u_2-u_3)}\\
&+ \frac{h(3-5u_2+5u_1(-1+2u_2))f_{n+u_3}}{60(u_1-u_3)(-1+u_3)(-u_2+u_3)},
\end{aligned}
$$

where, $f_{n+j} = f\,(t_{n+j}, v_{n+j})$, for $j = u_1, u_2, u_3, 1$, and $v_{n+j} \approx v(t_n + jh)$ are approximations of the exact solution. Expanding $v(t_{n+1})$ in the Taylor series around t_n gives the LTE.

$$
\begin{aligned}
\mathcal{L}(v(t_{n+1});h) = {}& \frac{1}{7200}(-2+3u_1+3u_2-5u_1u_2+3u_3-5u_1u_3-5u_2u_3\\
&+10u_1u_2u_3)y^6[t_n]h^6\\
&+\frac{1}{302400}(-24+21u_1+21u_1^2+21u_2-14u_1u_2-35u_1^2u_2+21u_1^2\\
&-35u_1u_1^2+21u_3)y^7[t_n]h^7 \qquad (11)\\
&+\frac{1}{302400}(-14u_1u_3-35u_1^2u_3-14u_2u_3+70u_1^2u_2u_3-35u_2^2u_3\\
&+70u_1u_2^2u_3)y^7[t_n]h^7\\
&+\frac{1}{302400}(21u_3^2-35u_1u_3^2-35u_2u_3^2+70u_1u_2u_3^2)y^7[t_n]h^7+O(h)^8.
\end{aligned}
$$

Equating the leading term of the LTE in equation (11) to zero yields:

$$
\frac{1}{7200}(-2+3u_1+3u_2-5u_1u_2+3u_3-5u_1u_3-5u_2u_3+10u_1u_2u_3)=0. \qquad (12)
$$

There are an infinite number of solutions for u_1, u_2, u_3 since there are more unknowns than equations. u_2 is optimized when two of them (let's say, u_1 and u_3) are thought of as free parameters. Adopting this approach in solving equation (12), one of the parameters is obtained in terms of the other two:

$$u_2 = \frac{2 - 3u_1 - 3u_3 + 5u_1u_3}{3 - 5u_1 - 5u_3 + 10u_1u_3}, \tag{13}$$

while the other two parameters are given as

$$u_1 = \frac{1}{10}\left(5 - \sqrt{5}\right); u_3 = \frac{1}{10}\left(5 + \sqrt{5}\right). \tag{14}$$

Substituting equation (14) into equation (13), we get $u_2 = \frac{1}{2}$.

2.1 Second derivative method with optimized hybrid points (SDMOHP)

Interpolating equation (2) at $t_{n+j}, j = 0$ and collocating equations (3) and (4) at $t_{n+j}, j = 0, u_1, u_2, u_3, 1$, and $t_{n+j}, j = 0, 1$ respectively yields a system of linear equations given as

$$\begin{pmatrix} 1 & t_n & t_n^2 & t_n^3 & t_n^4 & t_n^5 & t_n^6 & t_n^7 \\ 0 & 1 & 2t_n & 3t_n^2 & 4t_n^3 & 5t_n^4 & 6t_n^5 & 7t_n^6 \\ 0 & 1 & 2t_{n+u_1} & 3t_{n+u_1}^2 & 4t_{n+u_1}^3 & 5t_{n+u_1}^4 & 6t_{n+u_1}^5 & 7t_{n+u_1}^6 \\ 0 & 1 & 2t_{n+u_2} & 3t_{n+u_2}^2 & 4t_{n+u_2}^3 & 5t_{n+u_2}^4 & 6t_{n+u_2}^5 & 7t_{n+u_2}^6 \\ 0 & 1 & 2t_{n+u_3} & 3t_{n+u_3}^2 & 4t_{n+u_3}^3 & 5t_{n+u_3}^4 & 6t_{n+u_3}^5 & 7t_{n+u_3}^6 \\ 0 & 1 & 2t_{n+1} & 3t_{n+1}^2 & 4t_{n+1}^3 & 5t_{n+1}^4 & 6t_{n+1}^5 & 7t_{n+1}^6 \\ 0 & 0 & 2 & 6t_n & 12t_n^2 & 20t_n^3 & 30t_n^4 & 42t_{n+1}^5 \\ 0 & 0 & 2 & 6t_{n+1} & 12t_{n+1}^2 & 20t_{n+1}^3 & 30t_{n+1}^4 & 42t_{n+1}^5 \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \end{pmatrix} = \begin{pmatrix} v_n \\ f_n \\ f_{n+u_1} \\ f_{n+u_2} \\ f_{n+u_3} \\ f_{n+1} \\ g_n \\ g_{n+1} \end{pmatrix}. \tag{15}$$

Solving the system (16) by the Gaussian Elimination method, the coefficients w_j 's, $j = 0, 1, \ldots, 7$ are obtained and substituted into equation (2) gives an implicit continuous scheme of the form:

$$p(t) = \alpha_0(t)y_n + h(\beta_0(t)f_n + \beta_{u_1}(t)f_{n+u_1} + \beta_{u_2}(t)f_{n+u_2} + \beta_{u_3}(t)f_{n+u_3} + \beta_1(t)f_{n+1} + h^2(\gamma_0(t)g_n + \gamma_1(t)g_{n+1}). \tag{16}$$

Evaluating equation (15) at the points $t = t_{n+u_1}, t_{n+u_2}, t_{n+u_3}, t_{n+1}$, yield the following

$$\begin{aligned} v_{n+u_1} = v_n + \frac{1}{105000}\Big(h\Big(\left(14325 + 105\sqrt{5}\right)f_n + \left(15625 + 135\sqrt{5}\right)f_{n+u_1} + (8000 \\ - 6464\sqrt{5})f_{n+u_2} + \left(15625 - 5625\sqrt{5}\right)f_{n+u_3} + \left(-1075 + 107\sqrt{5}\right)f_{n+1}\Big) \\ + h^2\left(\left(600 + 8\sqrt{5}\right)g_n + \left(100 - 8\sqrt{5}\right)g_{n+1}\right)\Big), \\ v_{n+u_2} = v_n + \frac{1}{13440}\Big(h\Big(1723f_n + \left(2000 + 875\sqrt{5}\right)f_{n+u_1} + 1024f_{n+u_1} \\ + \left(2000 - 875\sqrt{5}\right)f_{n+u_3} - 27f_{n+1}\Big) + h^2(67g_n + 3g_{n+1})\Big), \end{aligned} \tag{17}$$

$$v_{n+u_3} = v_n + \frac{1}{105000}\Big(h\Big(\big(14325 - 105\sqrt{5}\big)f_n + \big(15625 + 135\sqrt{5}\big)f_{n+u_1} + (8000 + 6464\sqrt{5})f_{n+u_2} + \big(15625 - 5625\sqrt{5}\big)f_{n+u_3} - \big(1075 + 107\sqrt{5}\big)f_{n+1}\Big) + h^2\Big(\big(600 - 8\sqrt{5}\big)g_n + (100 + 8\sqrt{5})g_{n+1}\Big)\Big),$$

$$v_{n+1} = v_n + \frac{1}{420}\Big(h\big(53f_n + 125f_{n+u_1} + 64)f_{n+u_2} + 125f_{n+u_3} + 53f_{n+1}\big) + h^2(2g_n - 2g_{n+1})\Big).$$

3. Basic properties of the SDMOHP

This section examines the SDMOHP (17) for accuracy, consistency, zero-stability, convergence, linear stability, and A-stability.

3.1 Order of accuracy and consistency

The matrix difference form for the SDMOHP (17) is given as

$$W_1V_n = W_0V_{n-1} + h(Z_0F_{n-1} + Z_1F_n) + h^2(V_0G_{n-1} + V_1G_n), \tag{18}$$

Where W_0, W_1, Z_0, Z_1, V_0 and V_1 are 4×4 matrices given by

$$W_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix};\ W_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};\ Z_0 = \begin{pmatrix} 0 & 0 & 0 & \frac{14326+107\sqrt{5}}{105000} \\ 0 & 0 & 0 & \frac{1723}{13440} \\ 0 & 0 & 0 & \frac{14326-107\sqrt{5}}{105000} \\ 0 & 0 & 0 & \frac{53}{420} \end{pmatrix}; \tag{19}$$

$$Z_1 = \begin{pmatrix} \frac{15625 + 1375\sqrt{5}}{105000} & \frac{8000 - 6464\sqrt{5}}{105000} & \frac{15625 - 5625\sqrt{5}}{105000} & \frac{-1075 + 107\sqrt{5}}{105000} \\ \frac{2000 + 875\sqrt{5}}{13440} & \frac{1024}{13440} & \frac{2000 - 875\sqrt{5}}{13440} & \frac{-27}{13440} \\ \frac{15625 + 5625\sqrt{5}}{105000} & \frac{8000 + 6464\sqrt{5}}{105000} & \frac{15625 - 1375\sqrt{5}}{105000} & \frac{-(1075 + 107\sqrt{5})}{105000} \\ \frac{125}{420} & \frac{64}{420} & \frac{125}{420} & \frac{53}{420} \end{pmatrix}. \tag{20}$$

$$V_0 = \begin{pmatrix} 0 & 0 & 0 & \frac{600+8\sqrt{5}}{105000} \\ 0 & 0 & 0 & \frac{67}{13440} \\ 0 & 0 & 0 & \frac{600-8\sqrt{5}}{105000} \\ 0 & 0 & 0 & \frac{1}{210} \end{pmatrix}; V_1 = \begin{pmatrix} 0 & 0 & 0 & \frac{100-8\sqrt{5}}{105000} \\ 0 & 0 & 0 & \frac{3}{13440} \\ 0 & 0 & 0 & \frac{100+8\sqrt{5}}{105000} \\ 0 & 0 & 0 & \frac{-1}{210} \end{pmatrix}; \tag{21}$$

$$V_n = (v_{n+u_1}, v_{n+u_2}, v_{n+u_3}, v_{n+1})^T,$$

$$V_{n-1} = (v_{n-1+u_1}, v_{n-1+u_2}, v_{n-1+u_3}, v_n)^T,$$

$$F_n = (f_{n+u_1}, f_{n+u_2}, f_{n+u_3}, f_{n+1})^T,$$

$$F_{n-1} = (f_{n-1+u_1}, f_{n-1+u_2}, f_{n-1+u_3}, f_n)^T \tag{22}$$

$$G_n = (g_{n+u_1}, g_{n+u_2}, g_{n+u_3}, g_{n+1})^T,$$

$$G_{n-1} = (g_{n-1+u_1}, g_{n-1+u_2}, g_{n-1+u_3}, g_n)^T.$$

The SDMOHP is of order $p = (7,7,7,8)^T$ while the error constant is

$$c_{p+1} = \frac{-1}{75600000}, \frac{-1}{154828800}, \frac{-1}{75600000}, \frac{1}{1016064000}. \tag{23}$$

Hence, the SDMOHP has at least seventh-order accuracy.

3.2 Zero-stability and convergence

Zero-stability involves the characteristics exhibited by a procedure when the value of h approaches zero. In the context of a homogeneous equation $v' = 0$, the discretized form is

$$W_1 V_n - W_0 V_{n-1} = 0, \tag{24}$$

where W_0 and W_1 are given in equations (27) and (34). The first characteristic polynomial $\rho(\sigma) = det\,(\sigma W_1 - W_0) = \sigma^3(\sigma - 1) = 0$. This implies that $\sigma_1 = \sigma_2 = \sigma_3 = 0, \sigma_4 = 1$.

"A continuous implicit multistep method is zero-stable if the modulus of no root of the first characteristic polynomial $\rho(\sigma)$ exceeds one, and if every root with a modulus of one has a multiplicity that does not exceed the order of the differential equation" [7]. Therefore, the approach being presented exhibits zero stability.

Since the SDMOHP is both consistent and zero-stabile, it is convergent according to [10].

3.3 Linear stability

The concept of linear stability focuses on the performance of a method in real-world scenarios, where it is crucial to ascertain if the approach will produce desirable outcomes for a given positive value of h. To validate this concept, commonly known as linear stability, we employ the methodology on a linearized test problem.

$$v(t) = \mu v(t), Re(\mu) < 0. \tag{25}$$

Applying the SDMOHP to (25), yields the recurrence relation

$$V_n = H(\Lambda)V_{n-1}, \Lambda = \mu h, \tag{26}$$

where,

$$H(\Lambda) = (W_1 - \mu Z_1)^{-1}(W_0 - \mu Z_0), \tag{27}$$

is the stability matrix having eigenvalues $(\zeta_1, \zeta_2, \zeta_3, \zeta_4) = (0,0,0,\zeta_4)$ with

$$\zeta_4 = \frac{\Lambda^5 + 27\Lambda^4 + 360\Lambda^3 + 2820\Lambda^2 + 12600\Lambda + 25200}{-\Lambda^5 + 27\Lambda^4 - 360\Lambda^3 + 2820\Lambda^2 - 12600\Lambda + 25200}. \tag{28}$$

The stability region of SDMOHP is plotted in the complex plane as shown in Figure 1. The SDMOHP is A-stable since the whole left half-plane is included in the stability region.

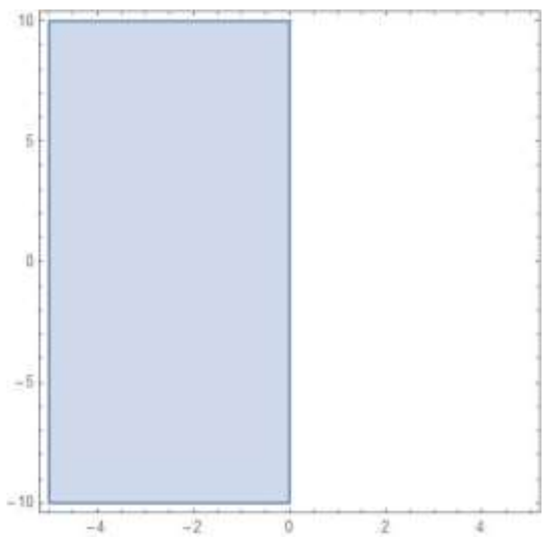

Figure 1: Region of absolute stability for SDMOHP

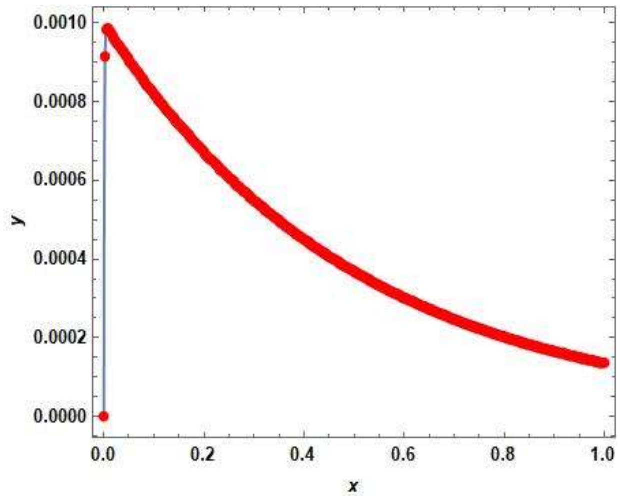

Figure 2: Solution plot for problem 4.1

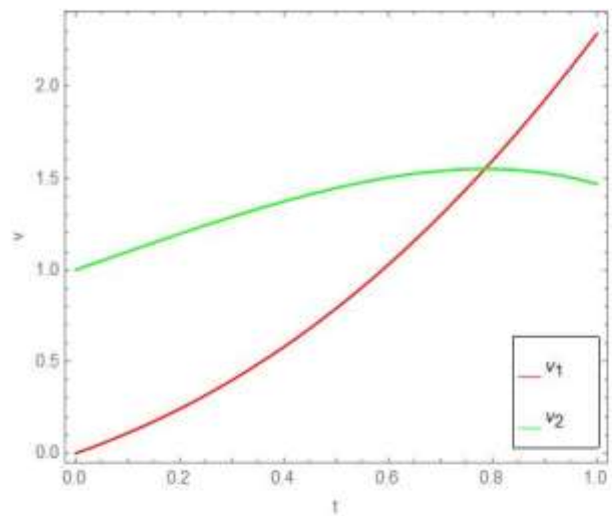

Figure 3: Solution plot for problem 4.2

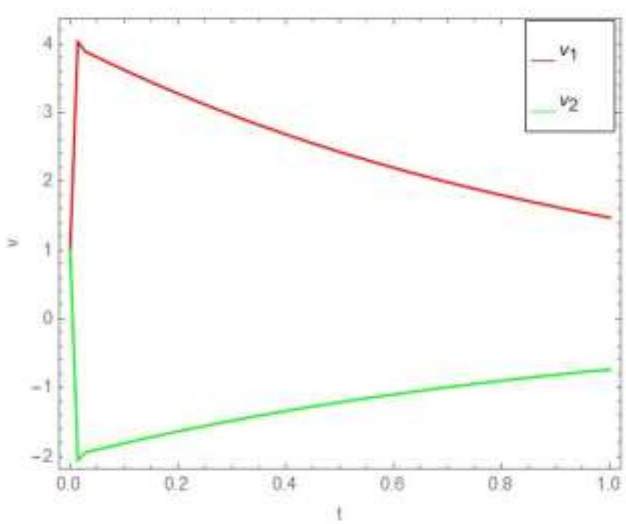

Figure 4: Solution plot for problem 4.3

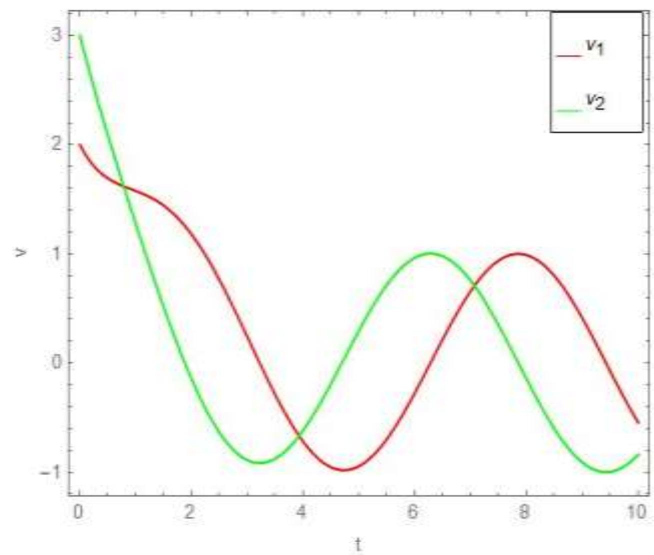

Figure 5: Solution plot for problem 4.4

4. Numerical experiments and results

The SDMOHP is applied to solve some first-order IVPs and comparison is made with existing methods in the literature. The selected methods are SDMOHP (17), the Hybrid Block Second Derivative Backward Differentiation Formula (HBSDBDF) in [2], and the reformulated two-step block optimized hybrid method (RBHMO) in [14].

The following are used to measure the performance of each of the aforementioned methods. The maximum global absolute error (MErr), global absolute error (AbErr), absolute error at and the final grid point (FErr). The problems used to test the accuracy of the schemes are the nearly sinusoidal system in [2], the linear stiff IVP considered in [14], the system of linear equations in [21], and the mildly stiff problem in [22].

Problem 4.1

Given the linear stiff IVP

$$v'(t) = -1000v + e^{-2t}, \;\; v(0) = 0. \tag{29}$$

The exact solution is $v(t) = \frac{1}{908}(e^{-2t} - e^{-1000})$. This equation has been subject to several numerical investigations in the literature such as [10]. The interval of solution is [0,1] with $n = 3750, 7500, 15000$. The AbErr is computed using the methods SDMOHP, HBSDBDF, and RBHMO, and the results are presented in Table 1. The solution profile is presented in Figure 2. The results in table indicate that the SDMOHP outperforms existing methods with respect to accuracy.

Problem 4.2

Given the system of linear IVP:

$$v_1'(t) = v_1(t) + v_2(t), \;\; v_1(0) = 0, \tag{30}$$

$$v_2'(t) = -v_1(t) + v_2(t), \;\; v_2(0) = 0.$$

The exact solution $v_1(t) = e^t \sin t, v_2(t) = e^t \cos t$. This problem was numerically investigated by [19]. The problem is solved for step sizes $n = 20, 40, 80$ with the MErr, FErr, and AbErr computed using the methods SDMOHP, HBSDBDF and RBHMO and results presented in Table 2. The solution profile is represented in Figure 3. The results in Table 2 reveal that the SDMOHP outperform existing methods concerning accuracy.

Problem 4.3

Given the mildly stiff problem with stiffness ratio 1:1000 investigated by [20] among others,

$$v_1'(t) = 998v_1(t) + 1998v_2(t), \;\; v_1(0) = 1,$$
$$v_2'(t) = -999v_1(t) - 1999v_2(t), \;\; v_2(0) = 0, \tag{31}$$

with exact solution

$$v_1(t) = 4e^{-t} - 3e^{-1000t},$$
$$v_2(t) = -2e^{-t} + 3e^{-1000t}. \tag{32}$$

The interval of solution is [0,1] with $n = 30, 50, 70$. The AbErr is computed using the methods SDMOHP, HBSDBDF and RBHMO, and results are presented in Tables 3. As revealed by Table 3, the SDMOHP is more accurate than the existing methods.

Problem 4.4

Given the nearly sinusoidal system investigated by [2]

$$v_1'(t) = -2v_1(t) + v_2(t) + 2\sin t, \;\; v_1(0) = 2,$$
$$v_2'(t) = 998v_1(t) - 999v_2(t) + 999\cos t - 999\sin t, \;\; v_2(0) = 3, \tag{33}$$

with exact solution

$$v_1(t) = 2e^{-t} + \sin t,$$
$$v_2(t) = 2e^{-t} + \cos t. \tag{34}$$

The problem is solved in the interval [0,1] taking $n = 25, 50, 100, 200$. The MErr, and Ferr, are computed using the methods SDMOHP, HBSDBDF and RBHMO, and results are presented in Tables 4. As revealed by Table 4, the SDMOHP outperforms existing methods with respect to accuracy.

Table 1: The AbErr for Problem 4.1 using different methods and step sizes (n)

n	Method	AbErr
100	SDMOHP	0.000E+00
	HBSDBDF	2.711E-20
	RBHMO	3.795E-19
200	SDMOHP	0.000E+00
	HBSDBDF	2.711E-20
	RBHMO	2.711E-20
400	SDMOHP	1.044E-14
	HBSDBDF	0.000E+00
	RBHMO	0.000E+00

Table 2: The MErr, and FErr for Problem 4.2 using different methods and step sizes (n)

n	Method	MErr-v_1	MErr- v_2	FErr- v_1	FErr- v_2
10	SDMOHP	7.10543E-15	1.55431E-15	7.10543E-15	8.88178E-16
	HBSDBDF	4.71915E-09	1.94067E-09	4.71915E-09	1.92155E-09
	RBHMO	1.66107E-08	6.83124E-09	1.66107E-08	6.09217E-09
20	SDMOHP	1.77636E-15	1.77636E-15	1.77636E-15	1.55431E-15
	HBSDBDF	3.22329E-11	1.67675E-11	3.22329E-11	1.67675E-11
	RBHMO	2.59210E-10	1.07343E-10	2.59210E-10	9.60974E-11
40	SDMOHP	6.21725E-15	4.88498E-15	6.21725E-15	4.66294E-15
	HBSDBDF	1.51434E-13	1.15685E-13	1.51434E-13	1.15685E-13
	RBHMO	4.05542E-12	1.67311E-12	4.05542E-12	1.49569E-12
80	SDMOHP	6.21725E-15	4.88498E-15	6.21725E-15	4.66294E-15
	HBSDBDF	3.41949E-14	6.17284E-14	9.76996E-15	5.30687E-14
	RBHMO	5.15143E-14	2.66454E-14	3.81917E-14	2.66454E-14

Table 3: The AbErr for Problem 4.3 using different methods and step sizes (n)

n	Method	AbErr- v_1	AbErr- v_2
30	SDMOHP	6.661E-16	2.220E-16
	HBSDBDF	1.146E-12	5.729E-13
	RBHMO	2.846E-07	2.846E-07
50	SDMOHP	5.329E-15	2.554E-15
	HBSDBDF	3.439E-13	1.720E-13
	RBHMO	1.554E-14	7.827E-15
70	SDMOHP	1.044E-14	5.107E-15
	HBSDBDF	3.124E-13	1.561E-13
	RBHMO	1.721E-14	8.604E-15

Table 4: The MErr for Problem 4.4 using different methods and step sizes (n)

n	Method	MErr- v_1	MErr- v_2
25	SDMOHP	7.17293E-12	7.16449E-12
	HBSDBDF	1.38968E-06	3.84999E-05
	RBHMO	3.81823E-07	4.19753E-07
50	SDMOHP	2.79776E-14	2.78944E-14
	HBSDBDF	1.32072E-08	1.32079E-08
	RBHMO	5.97340E-09	8.37251E-09
100	SDMOHP	2.99760E-15	2.99760E-15
	HBSDBDF	1.13631E-10	1.13635E-10
	RBHMO	9.32286E-11	1.89637E-10

5. Conclusion

We have presented the seventh-order second derivative method with optimized hybrid points for solving first-order initial value problems of ODEs. The results in Tables 1, 2, 3, and 4 reveal that the methods SDMOHP is more accurate than existing methods. In comparison with another popular method from available literature, our methods produced minor errors. The SDMOHP was implemented in block modes with the merit that no starting values were required. The methods have good accuracy properties and are indeed of the higher order of accuracy at the final grid point where the LTE was optimized, a major advantage of the method. Also, the methods do not require the creation of separate predictors. Hence, the technique is recommended for general use.

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Conflict of interest

The authors declare that there is no competing interest.

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