On the Performance Measure for Customers' Intermittencies in Vehicle Routing Problems with Priorities

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Abstract: This paper discusses variant categories of Late Request Customers (LRC) as they interpose with Early Request Customers (ERC) thereby resulting in Intermittent Vehicle Routing Problems (IVRP). We conceptualize the multiple priorities that arise in vehicle routing and the connectivity between priorities based on Time and Quantity as it affects the interjectory intermittent dynamical situations. Solving problems of this nature could be quite challenging. It requires optimization along different dimensions and directions. This is connected to the fact that uncertainties associated with real-life situations make life dynamic hence opening the vista that brought about intermittencies in Vehicle Routing Problems (IVRP). To achieve these feats, this paper formulates a relation that fuses the various categories of LRC into existing ERC, analyses the fused LRC, and encapsulates priorities into the formulated relation. The formulated relation embraces a stochastic process introduced to the existing deterministic process, leading to dynamic situations in a typical VRP setting as they occur in real life.

Index Terms: Early Request Customers (ERC), Late Request Customers (LRC), Intermittent Situation, Re-optimization, Re-activation, Multiple Priorities.

I. INTRODUCTION

In practice, the objective function of Intermittencies in Vehicle Routing Problems (IVRP) with priorities can be quite complex to categorically formulate for so many things are involved. Such complexity could include the minimization of the difference between the farthest and shortest routes; balancing of the workload between/among the drivers, minimizing the number of vehicles hence, reducing overhead costs, satisfying the priorities, and/or maximization the number of customers to be serviced to improve the service level.

In some instances, the robustness of the solution is one of the important objectives. However, these various objectives more often may conflict with one another hence, we shall be looking at how best IVRP with multiple priorities can be modeled and be solved despite the request times and locations that follow a probability scheme.

Here, according to [1] and [2], the initially known customers before the vehicle set out from the depot are referred to as Early Request Customers (ERC) with

$$ERC = \sum_{i=1}^{N} c_i \qquad (1)$$

where N is the number of customers in the set, $c_i = \{c_1, c_2, \ldots, c_N\}$, of customers. The stochastically requesting customers are referred to as Late Request Customers (LRC) with

$$LRC = \sum_{i=1}^{x} c_{N_S}^{i} \qquad (2)$$

where the number of anticipatory customers is x, customers already being serviced before the LRC is to be served is represented by N_S, and $c_{N_S}^{i}$ is the set of LRC.

Within a frame of time, every vehicle is expected to reach the customers at a service point. The entire tour starts at the depot and should ultimately terminate at the depot. The ERC must be served. In the event of the tour, a new set of $LRCs$ may stochastically request service. The LRC remains not known except when their requests are placed. The LRC requests time and location tolls from a known probability scheme.

Each time a vehicle sets out to service customers, the dispatcher decides the following:

(a) which of the occurred requests' subset is to be designated to a particular vehicle;

(b) a report is kept on the entire tour as long as the vehicle is still within the service period.

When a vehicle is assigned to the LRC, the vehicle is expected to service the remaining customer in that route within the tour time frame. The dispatcher aims to maximize the number of $LRCs$ to which a vehicle is assigned subject to the degree of dynamism. Depending on the number of $ERCs$ and their corresponding arrival time, the authors in [3] defined and suggested an effective way to measure the degree of dynamism. Also, since the problem is intermittently dynamic, it is necessary to adaptively modify the existing service pattern to serve the LRC to adhere to the priorities.

The most efficient means to attain this is to modify part of the ERC solution then, the LRC has to be infiltrated into the existing ERC under consideration. For better planning, it is wise to take into consideration future customers for several reasons. Solving a dynamical pickup and delivery problem opined by the authors in [4] suggested a double-horizon heuristic that focuses on both short-term goals by minimizing the total distance traveled and long-term goals by maximizing the slack time to accommodate servicing of the LRC. However, [5], [6], and [7] have investigated the waiting strategies and improved on the solution by making vehicles wait at certain places to buy more time.

II. CATEGORIES OF DYNAMICAL DEGREE (DD)

The Intermittent Vehicle Routing Problems (IVRP) with stochastic requests differ in their levels of uncertainty. Particularly, it differs on how many ERCs are known before takeoff and how many LRCs might place orders within the time horizon.

The authors in [1], [2], and [8] represented the rate of the uncertainty of customers' expected requests as the Dynamical Degree or Degree of Dynamism (DD). The relation for DD is defined as the ratio of the numbers of LRC to the Overall Customers (OC):

$$DD = \frac{\text{LRC}}{\text{OC}} \qquad (3)$$

The Overall Customers (OC), which is the total numbers of ERC and LRC to be considered have three possible ways of occurring:

(i) In the first kind, as presented by [2], the LRC comes after all the ERC orders have been met. The relations (1) and (2) thus give rise to:

$$\text{OC} = \text{ERC} + \text{LRC} = \sum_{i=1}^{N} c_i + \sum_{i=1}^{x} c_{N_S}^{i} \qquad (4)$$

where N is the number of ERCs, the number of $LRCs$ is x and N_S is the number of customers in ERC that have been attended to.

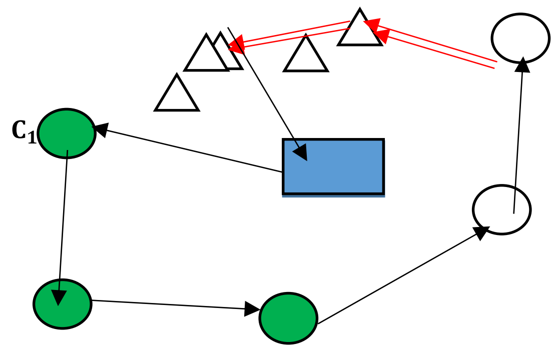

Figure 1: LRC Pre-Decision State

This first category is depicted in Figure 1 and classified as Pre-Decision State. The case is usually easy to address relative to other classifications in that, the ERC initially planned tour is not affected in any way. With or without the LRC, the ERC is kept intact and all the customers in this category are treated and served first. It gives room for anticipatory demand planned alongside the tour. The DD from (3) and (4) in this case is given by:

$$\text{DD} = \frac{\text{LRC}}{\text{OC}} = \frac{\text{LRC}}{\text{ERC+LRC}} = \frac{\sum_{i=1}^{x} c_N^i}{\sum_{i=1}^{N} c_i + \sum_{i=1}^{x} c_{N_S}^i} \qquad (5)$$

Similar to (5) is observed in [9] and [10].

(ii) Second, is a case where the $LRCs$ come after a fractional part of the ERC, the order has been met $i.e. ERC(1)$, and the remaining ERC i.e. $ERC(2)$ is given by

$$ERC(2) = ERC - ERC(1) \qquad (6)$$

ERC(2) orders are met after all possible $LRCs$ have been serviced thus:

$$\text{OC} = \text{ERC}(1) + \text{LRC} + \text{ERC}(2) \qquad (7)$$

where $ERC(1) + ERC(2) = ERC$ and the resulting OC in the second case is given by:

$$OC = \sum_{i=1}^{G_1} c_i + \sum_{i=1}^{x} c_{N_S}^i + \sum_{i=G_1+1}^{N-G_1} c_i \qquad (8)$$

where $G_1 < N$, represents the customer(s) in turn in the ERC that have just been serviced after which the LRC demand is to be met and $x \leq k$ represents the maximum possible LRC that can be taken into consideration such that the initial tour plan will not be uttered or affected.

Also, from (3) and (8), the DD in the second case is given by:

$$\text{DD} = \frac{\text{LRC}}{\text{OC}} = \frac{\text{LRC}}{\text{ERC(1)+ LRC + ERC(2)}} \qquad (9)$$

$$DD = \frac{\sum_{i=1}^{x} c_{N_S}^i}{\sum_{i=1}^{G_1} c_i + \sum_{i=1}^{x} c_{N_S}^i + \sum_{i=G_1+1}^{N-G_1} c_i} \qquad (10)$$

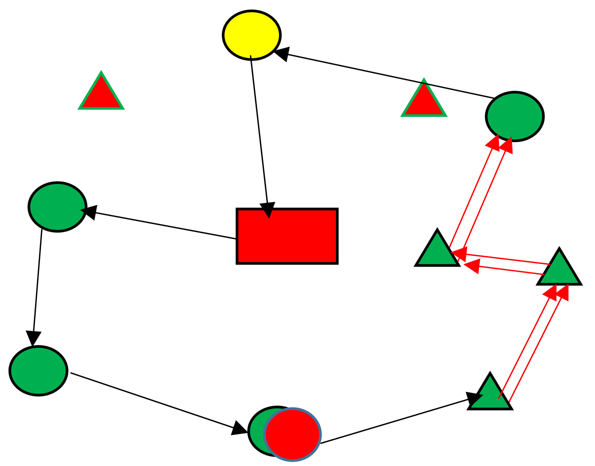

Figure 2: LRC Nick-Decision State

The second category is shown in Figure 2. Figure 2 depicts the case in (10) in which the LRC comes between ERC just once.

(iii) The third case is a situation where the LRC comes intermittently within the $ERCs$ with a proviso that the LRC can't come before the first ERC i.e.,

$$LRC = \sum_{i=1}^{x_1} c_{G_1}^i + \sum_{i=1}^{x_2} c_{G_2}^i + \cdots + \sum_{i=1}^{x_{n-1}} c_{G_{n-1}}^i + \sum_{i=1}^{x_n} c_{G_n}^i \quad (11)$$

where $x_1, x_2, \dots, x_{n-1}, x_n$ represent various numbers of anticipatory customers coming up intermittently, and the corresponding OC is given by:

$$OC = ERC(1) + LRC(1) + ERC(2) + LRC(2) + \cdots + LRC(y) + ERC(M) \quad (12)$$

where $M \leq (N-1)$ is the possible number of ERC intermittencies serviced in the course of the servicing and $y \leq (x-1)$ is the number of LRC intermittencies undertaken. With

$$ERC(1) + \cdots + ERC(M) = ERC \quad (13)$$

and

$$LRC(1) + \cdots + LRC(y) = LRC. \quad (14)$$

The resulting OC in the third case is thus:

$$OC = \sum_{i=1}^{G_1} c_i + \sum_{i=1}^{x_1} c_{G_1}^i + \sum_{i=G_1+1}^{G_2=N-G_1} c_i + \sum_{i=1}^{x_2} c_{G_2}^i + \sum_{i=G_2+1}^{G_3=N-G_2} c_i + \sum_{i=1}^{x_3} c_{G_3}^i + \cdots + \sum_{i=1}^{x_n} c_{G_n}^i + \sum_{i=G_n+1}^{N-(G_1+G_2+\dots+G_n)} c_i \quad (15)$$

where $G_1, G_2, \dots, G_{n-1}, G_n$ represent the customers that had been serviced at the time the anticipatory customer's request comes in, and $G_1 + G_2 + \cdots + G_n = N$. Hence, the DD for the third case from (3), (11), and (15) is given by:

$$DD = \frac{\sum_{i=1}^{x_1} c_{G_1}^i + \sum_{i=1}^{x_2} c_{G_2}^i + \dots + \sum_{i=1}^{x_n} c_{G_n}^i}{\sum_{i=1}^{G_1} c_i + \sum_{i=1}^{x_1} c_{G_1}^i + \sum_{i=G_1+1}^{G_2=N-G_1} c_i + \sum_{i=1}^{x_2} c_{G_2}^i + \sum_{i=G_2+1}^{G_3=N-G_2} c_i + \sum_{i=1}^{x_3} c_{G_3}^i + \dots + \sum_{i=1}^{x_n} c_{G_n}^i + \sum_{i=G_n+1}^{N-(G_1+G_2+\dots+G_n)} c_i} \quad (16)$$

It is worth noting that, in any of the three cases, the LRC cannot come ahead of the first ERC. Otherwise, such an LRC automatically becomes an ERC.

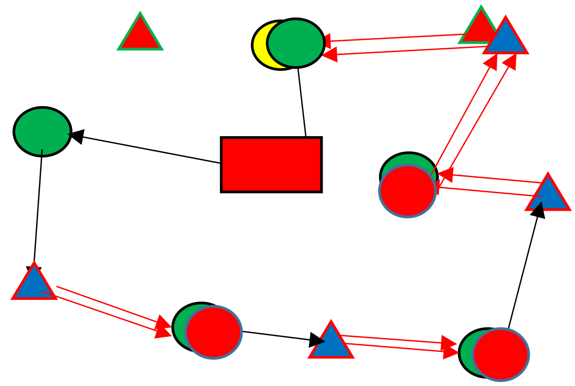

Figure 3: LRC Post-Decision State

The DD is a yardstick to classify IVRP into stochastic requests.

From [11] and [12], a moderate DD realized in practices includes the distribution of oils, transportation of patients, and grocery deliveries. Its scope of applications with high-level DD includes emergency vehicles or courier services as in [13] and [14]. A high-level DD is mostly found in practical applications which include: same-day delivery; responsive demand transportation and shared mobility as opined in [15], [16], and [17]. For a more detailed classification of DSVRP applications, see [2] and [18].

III. MARKOVIAN DECISION PROCESS MODEL (MDPM)

Here, the Intermittent VRP has to be modeled bearing in mind its stochastic nature and be treated alike. In [19], a model of the Markov Decision Process Model (MDPM) was built and constructed. The MDPM is split into three stages as in [2] thus:

Stage 1: Compute the LRC Pre-Decision State, S_k

The following notations are declared and used: Let the Pre-Decision State whenever the vehicle serves a customer be k out of the total N. The Pre-Decision State, S_k, is given by the relation:

$$S_k = (t_k, P_k, c_k, \theta_k, c_k^{new}) \quad (17)$$

consisting of a Point in Time, t_k, the location of the vehicle at the point in time, P_k. Let the number of customers not yet served in ERC be c_k. Assign to the tour plan sequence of LRC, (new requests), c_k^{new}, and let the Updated Planned Tour for ERC be given as:

$$\theta_k = (P_k, c_k^1, c_k^2, c_k^3, \dots, c_k^{N-x}, c_0) \quad (18)$$

where c_0 is the depot. Furthermore, the Pre-Decision State, S_k be the set of new requests, c_k^{new}, defined by:

$$c_k^{new} = \{c_k^1, c_k^2, c_k^3, \dots, c_k^l\} \quad (19)$$

such that l are random variables that represent the number of LRC requests.

Stage 2: Compute the LRC Nick-Decisions State, x

Nick-Decisions, x, are made as new sets of LRC to be served, $c_k^{x,\ assign}$ of c_k^{new}, that is assigned to lead to a set of updated customers to be serviced thus:

$$c_k^x = c_k \cup c_k^{x,\ assign} \quad (20)$$

The Decision to further service the LRC calls for the update of the tour from θ_k to θ_k^x following the sequences, c_k^x, and determines the next customer to be serviced. Here, Decisions are considered feasible provided the tour allows all the customers to be serviced as planned within the time frame, and the vehicle ultimately returns to the depot. The number of newly assigned LRC is the immediate Reward of Decision, x, in State:

$$S_k = \mathrm{R}(S_k, x) \quad (21)$$

The Decision transition between the states can be broken into (i) the Pre-Decision transition and (ii) the Post-Decision transition state. Likewise, the stochastic transition from a Post-Decision state changes to a new decision state.

Stage 3: Compute the LRC Post-Decision State, S_k^x

The Post-Decision state is given as:

$$S_k^x = (t_k, c_k^x, \theta_k^x) \quad (22)$$

consisting of the point in time, t_k, the customers that are yet to be served, c_k^x, and the planned tour, θ_k^x. The stochastic transition results from the total distance traveled by the vehicle, its services to the next customer, and a new stochastic set of LRC. The stochastic transition is aimed at updating the point a vehicle is at a particular time, the customers not yet served, and the set of newly arrived LRCs. The new decision state, S_{k+1}, combines the decision as well as the stochastic transition state. The process of the Post-Decision State terminates in the new decision state, S_{k+1}, as the limit time elapses and the vehicle is returned to the depot.

IV. IVRP Classifications

The classifications below show a situation in which a vehicle leaves the depot, serves as many customers as possible, and returns to the depot which is a complete cycle.

(a) Figure 1 represents the classification of the Pre-Decision State where the Green Circles indicate the customers that have been serviced, and the White Circles represent the customers that have not been serviced (rather than waiting in line to be served). The Triangles represent the LRC while the Rectangle represents the Depot.

(b) From (12), the Pre-Decision State is given as:

$$S_k = (540, P_k, \{c_1, c_2, c_3, c_4, c_5\}, (P_k, c_k^1, c_k^2, \dots, c_k^{N-x}, c_0), \{c_k^1, c_k^2, c_k^3, \dots, c_k^l\}) \quad (23)$$

The time, $t_k = 540$minutes, is depicted in Figure 1 with Five customers: c_1, c_2, c_3, c_4, and c_5, in succession following the direction of the arrow from the Depot, were assigned at the start of the tour. The planned tour, θ_k, is as indicated by the single arrows in Figure 1. The vehicle tour starts at c_0 and serves the ERC in this order c_1, c_2, c_3, c_4, and c_5 then, returns to the depot, c_0.

While this paper aims to stress the Intermittent in VRP, five new $LRCs$ placed orders requesting services after the tour commenced thus:

$$c_k^{new} = \{c_k^1, c_k^2, c_k^3, c_k^4, c_k^5\} \quad (24)$$

In Figure 2, since the request was placed after the third Green Circle, i.e. after the third customer then, $k = 3$. The vehicle has just serviced some customers and the current position of the vehicle which is a co-joined Green and Red Circles, means that the customer has been serviced but the tour needs to be re-optimized due to an LRC just coming in. The Yellow Circle indicates the readiness to return to the depot and the Red Rectangle indicates the return of the vehicle to the depot.

Then, the LRC is given as:

$$\mathrm{c_k^{new}} = \{\mathrm{c_3^1, c_3^2, c_3^3, c_3^4, c_3^5}\} \quad (25)$$

Hence, the Nick-Decision

$$x = (\mathrm{c_3^4, c_3^5, (P_3, c_5, c_4, c_3^3, c_3^2, c_3^1, c_3, c_2, c_1, c_0)}) \quad (26)$$

determines the assignment of customer

$$\mathrm{c_3^{5,assign}} = \{\mathrm{c_3^1, c_3^2, c_3^3}\} \quad (27)$$

It must be noted that, although there are five $LRCs$, only three could be serviced. This can be traced to the fact that; the vehicle has exhausted the carrying capacity or the due time to return to the depot has elapsed.

Then, updating the tour in (18) by considering the ERC and $LRC's$ sequence gives:

$$\theta_3^5 = (P_3, c_5, c_4, c_3^3, c_3^2, c_3^1\ c_3, c_2, c_1, c_0) \quad (28)$$

changing the previous sequence of servicing the customers to now include both the ERC and LRC. The next locations to be serviced are $\mathrm{c_3^1, c_3^2}$, and $\mathrm{c_3^3}$ indicated by the two arrows in Figure 2. Arriving at $\mathrm{c_3^1, c_3^2}$, and $\mathrm{c_3^3}$, a new set of requests that leads to the next state, $\mathrm{S_{k+1}}$, is revealed.

The number of newly assigned LRC as in (21) which implies the immediate reward gives rise to:

$R(S_3, 5) = 3.$ (29)

Leading to a Post-Decision State which is the application of the LRC Nick- Decision, x, thus:

S_3^5=(540, {c_1, c_2, c_3, c_4, c_5}
($P_3, c_5, c_4, c_3^3, c_3^2, c_3^1, c_3, c_2, c_1, c_0$)) (30)

Figure 2 is an example of a single intermittent in VRP with only one un-capacitated vehicle used to service both advanced and immediate request customers without any priorities placed on the LRC.

(c) Figure 3 gives a holistic view of IVRP with multiple LRCs. This is usually a very complex situation to handle as it involves several LRCs. The good part of it is that bringing in the LRC increases the customers thereby increasing the turnover and eventually leading to the expanse of business. In a normal setting, the LRC could be integrated into the ERC's planned routes and the service order of the ERC that has not been visited remains unchanged. This could be achieved with minimal delay bringing about a detour. This was illustrated in Figures 2 and 3 where servicing of the LRC leads to a large detour in the whole system which occurred after ERC in c_3 in Figure 2 and ERC in $c_2, c_3,$ and c_4 respectively in Figure 3.

In real-life situations, the inclusion of LRC leads to a much more challenging task which will necessitate a re-planning of the route systems that have not been visited in the route.

V. RE-ACTIVATION AND RE-OPTIMIZATION OF THE CLASSICAL VRP

The IVRP is grouped into classes depending on the nature and time of entering the LRC into the routing process. An exception to all the aforementioned classes is the complete ERC cycle. The class shows a situation in which the vehicle leaves the depot, c_0, and serves the ERC in the order $c_1, c_2, \ldots, c_N$ returns to the depot, c_0 without serving any LRC. This is a typical VRP. This class of IVRP without the LRC is regarded as a complete ERC cycle. The complete ERC cycle is a VRP process without re-activation or re-optimization.

A typical VRP process is such that the vehicle departs the depot, services as many customers as possible, and ultimately returns to the depot. However, whenever there is a truncation in the usual VRP process as a result of bringing in the LRC after the LRC has been attended to and the system has to go back to normal, the process of entering back into the system is called re-activation in the VRP process. When a system enters the re-activation process, the system begins to work normally as if no LRC ever enters the system.

The process of switching the servicing vehicle from the ERC to the LRC and later back to the ERC thus causes the entire planned optimization process by the dispatched manager to be uttered. To reinstate the initial set out, the dispatch manager has to invoke the re-optimization process. The re-optimization process in IVRP is a situation in which the initially set out dispatch timing, quantity allocation, and variable costing are readjusted to ensure that these factors are kept intact despite the re-activation. In real-life situations, the inclusion of LRC leads to a much more challenging task which will necessitate a re-planning of the route systems that are yet to be visited in the route.

Generally, the more the restriction and complexity of the routing problems, the more complex the inclusion of intermittent customers will be. For example, the inclusion of LRC in a time frame-constrained routing problem will be more tasking than its inclusion in a non-time window-constrained problem. It must be noted that, in an online routing system, the services to customers may be declined service if it is not possible to find a feasible bring such customers into the existing system.

Often, the policy of turning down customers for the reason(s) of time horizon, quantity demanded or the vehicle carrying capacity includes the order to service the customers on the following day, next trip, or get delivery being transferred to another vehicle. While in some systems such as a pick-up of long-distance courier mail, the service provider or the distributor might choose to transfer the customer to a close competitor where they are not able to service the customer rather than being blacklisted.

VI. VRP PRIORITIES

In recent times, it is clear that each customer places different priorities on their orders which are to be supplied by delivery vehicles. Such priorities include the time of delivery and the quantity to be supplied. The vehicles change over time due to the heterogeneity of the vehicle and splitting of deliveries to mention a few.

This work focuses mainly on formulating an objective function for IVRP that will imbed the priority based on time and priority based on quantity for the goods to be delivered. Hence, the next section will dwell on the priorities.

PRIORITY BASED ON TIME

The VRP extension that includes Time Windows was mentioned in [20] in which multiple periods were considered. It assumes that each customer will be visited subject to a time proposed by the customer and the feasible combinations of visiting periods of other customers. This is referred to as Vehicle Routing Problem with Time Window (VRPTW).

The VRPTW has been applied to solve various real-life situations in VRP which include bus routing planning of [21], industrial waste collection observed by [22], where [23] and [24] stressed home delivery and petrol station replenishment of worked on by [25]. Others include bank, postal, and restaurant deliveries, national franchise, and security patrol services to mention but a few of the list. Lately, the applications of VRPTP include routing and scheduling of preventive maintenance of elevator service teams at customer locations [26] and blood products periodic delivery to hospitals by the Austrian Red Cross [27].

The authors in [1] and [2] enumerated some time windows to be considered herein. Each customer, c_i, has time windows that fall within $t_n^e < t_i < t_n^l$, i.e. an interval $(\boldsymbol{t^e}, \boldsymbol{t^l})$, (See [28], corresponding to the earliest time and latest time respectively that a vehicle visits the customer, c_i.

Let t_i represent the time required to service the customer c_i then, the following scenarios arise as stated in [2] thus:

$\boldsymbol{PT_1 = (t^e, t^l)}$: In this time sub-division, the vehicle could get to the customer's location any time and could depart at any time provided it is before the latest time. It implies that the customer should expect the vehicle at any time within the working hour and the vehicle will depart at any time only if the delivery has been done before the latest time required by the customer. Of all the time window sub-divisions, $(\boldsymbol{t^e}, \boldsymbol{t^l})$ enables the vehicle to service the customer at a convenient time within the working period, $\boldsymbol{T_k}$, thereby increasing the chances of visiting other customers with a tight time window.

$\boldsymbol{PT_2 = (t^e, t^l]}$: Herein, the time frame gives room for the vehicle to get to the location of the customer at any time as much as it falls within the working period but such vehicle must leave the customer on or before the latest departure time. This allows the vehicle to visit other customers before visiting the said customer.

$\boldsymbol{PT_3 = [t^e, t^l)}$: In this sub-division, the vehicle has been fixed to arrive at a specified time. The specified time must be at the nick of the earliest time or after the earliest time but could leave any time in as much as it is not beyond the latest departure time. The arrival time is closed while the departure time is open. In this case, if the vehicle arrives before the earliest set time, it will not be allowed to unload hence, it has to tarry till the earliest arrival time. This time frame is a little rigid.

$\boldsymbol{PT_4 = [t^e, t^l]}$: This time sub-division allows vehicles to arrive earlier than the earliest time, $\boldsymbol{t^e}$, however, the vehicle has to tarry in discharging till the earliest time, $\boldsymbol{t^e}$ before it can serve the customer. Though the vehicle is not allowed to discharge ahead of the earliest time, it must ultimately leave at $\boldsymbol{t^l}$ or before the latest time. This case is restricted at both the earliest and latest time. It is not flexible enough to allow for vehicles to come in at will ahead of the earliest time and depart the same on or before the latest departure time. In cases at the time set for the latest departure where the vehicle has not finished discharging, the vehicle has to leave to give room for other things. This case does not allow the unloading time to be elongated longer than necessary for the customer might have other schedules of activities to be attended to.
Summarily, from Figure 4, every customer, c_i, can only be linked to a time priority as:

$$\mu(c_i) = PT_1 \ or \ PT_2 \ or \ PT_3 \ or \ PT_4 \qquad (31)$$

where $\mu(c_i)$ represents the priority of a customer.

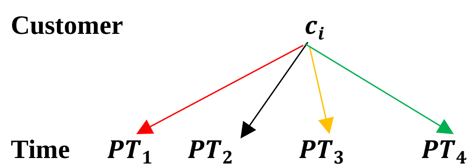

Fig. 4: Time Priorities Formulation Tree

PRIORITY BASED ON QUANTITY

Here, two important priorities based on quantity cases to be considered are opined by [1], thus:
(i) a situation where the quantity required by the customer is less than the carrying capacity of the vehicle. The quest for servicing customers on that route and meeting the specified quantities without a tradeoff is a key factor to be noted.
(ii) a situation in which the quantity demanded by a customer cannot be satisfied by one vehicle. This situation usually happens when the demand of the customer exceeds the vehicle carrying capacity. On the side of the supplier, this situation can turn out to be cost-effective in that, the vehicle can move straight to the customer's location.

Every customer, c_i, has a quantity demanded, i.e. that falls within the interval $[\boldsymbol{q^{min}}, \boldsymbol{q^{max}}]$, this is the minimum and maximum quantities demanded by the customer, c_i, respectively (see [2] and [28]). Other variants of the priority based on quantity are thus:

$\boldsymbol{PQ_1} = (\boldsymbol{q^{min}}, \boldsymbol{q^{max}})$: This indicates that the quantities required by the customer are not closed at both ends. This implies that the customers have neither a minimum nor a maximum quantity to deliver. This leaves the vehicle with the option to deliver as much product as possible to the customer as long as it is achieved within the vehicle's carrying capacity. The flexibility of this sub-division makes it difficult for the vehicle to properly plan the quantities to be delivered to each customer before leaving the depot.

$\boldsymbol{PQ_2} = (\boldsymbol{q^{min}}, \boldsymbol{q^{max}}]$: Here, there is a maximum quantity of products, goods, or services to which the customer is looking forward to receiving. Even though the supplier can or is ready to give more, the customer will not go beyond that maximum quantity for one reason or the other. However, the sub-division makes room for a supply of less.

$\boldsymbol{PQ_3} = [\boldsymbol{q^{min}}, \boldsymbol{q^{max}})$: Here, the customer has a predetermined quantity such that the amounts to be delivered cannot be less than the predetermined amount but, could be more than the predetermined quantity provided the vehicle can deliver it. In practical terms, this allows for the customers to accept more quantities higher than the requested quantity earlier at the point of delivery. This would not allow the vehicle to deliver all that the customers have been penciled for service on that route.

$\boldsymbol{PQ_4} = [\boldsymbol{q^{min}}, \boldsymbol{q^{max}}]$: In this case, the customer has a confined range of quantities to be delivered which cannot be more or less than the quantity required. It does not allow for the customer to increase the demand that has earlier been requested. It is stickily closed below and above. In practice, there could be situations where a customer might need more goods at the point of delivery owing to a pressing request not earlier envisaged owing to patronage. This restriction does not give room for flexibility on the part of the customer. As such, the supplier is at a disadvantage. It is worthy of note that, in practical situations where the quantity interval is closed at both ends, it is not flexible enough to make modifications. This is not an ideal priority setting as it gives no room for future supplier expansion and business growth.

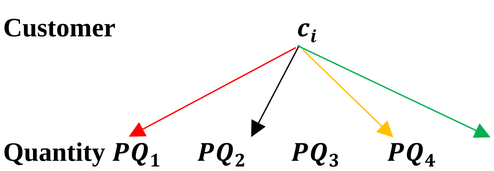

Figure 5: Quantity Priorities Formulation Tree

From Figure 5, each customer, c_i, can only be linked to a quantity priority such that:

$$\theta(c_i) = PQ_1 \; or \; PQ_2 \; or \; PQ_3 \; or \; PQ_4 \qquad (32)$$

where $\theta(c_i)$ represents the priority based on the quantity of a customer.

It is worthy of note that, every customer has priorities based on time and quantity. Hence, each customer must fulfill only one of the priorities based on time and quantity. Next, we move to priority formulations since only a set of time and quantity priority conditions can be fulfilled by each customer thus:

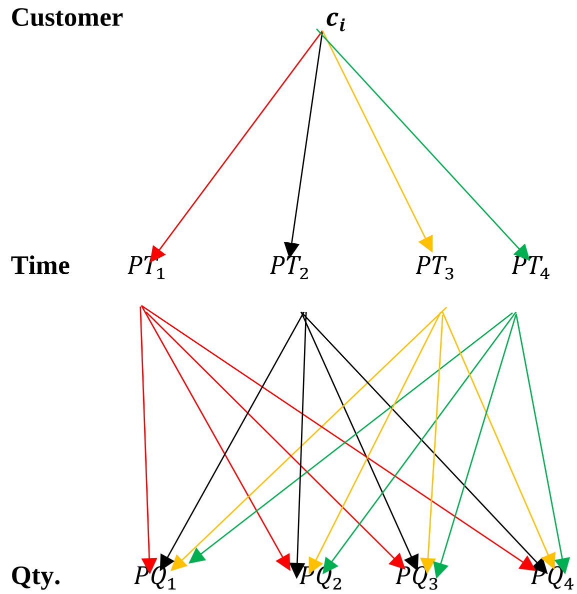

Figure 6: Priorities Formulation Tree

Figure 6 shows an interplay between the priorities. When the customer, c_i, is not serviced, the Time Priority of such customer, $PT_n = 0$, with $1 \leq n \leq 4$. Also, if a customer, c_i, is not serviced then, the quantity, $PQ_n = 0$ else, $PT_n = 1$ and $PQ_n = q_n$. Each customer fulfills a time and a quantity priority at a given time. The inter-relationship between the priorities for the customers is given by:

$$\delta(c_i) = \mu(c_i) + \theta(c_i) =$$
$$[PT_1(PQ_1) \; or \; PT_1(PQ_2) \; or \; PT_1(PQ_3) \; or \; PT_1(PQ_4)]$$

$$[PT_2(PQ_1) \; or \; PT_2(PQ_2) \; or \; PT_2(PQ_3) \; or \; PT_2(PQ_4)]$$

$$[PT_3(PQ_1) \; or \; PT_3(PQ_2) \; or \; PT_3(PQ_3) \; or \; PT_3(PQ_4)]$$

Or

$$[PT_4(PQ_1)\ or\ PT_4(PQ_2)\ or\ PT_4(PQ_3)\ or\ PT_4(PQ_4)] \quad (33)$$

The interplay resulting from priorities for all the customers is given by:

$$\sum_{i=1}^{N} \delta(c_i) = \sum_{i=1}^{N} \mu(c_i) \sum_{i=1}^{N} \theta(c_i)\} \quad (34)$$

This shows that, for a particular time window a customer chooses, other time windows not chosen are zero. Whichever time window a customer has chosen will ultimately be associated with the customer's desired quantity priority. However, when a particular quantity priority is chosen, the other remaining quantity priorities that are not chosen become zero. This eventually leaves the option that, only one particular time window interplays with one particular quantity priority for each of the customers.

The next section focuses on the inclusion of the priorities into the objective function formulation.

VII. PROBLEM FORMULATION

Considering the vehicle routing for a given period, T. Let $C = \{c_i: i = 0, 1, 2, \dots, N\}$ be the set of N customers with c_0 as the depot. Let $V = \{v_h \mid h = 1, 2, 3, \dots, M\}$ represent the set of M homogenous vehicles that are stationed at the depot, c_0. By [29], (i, j) is the associated pair of locations with $i,\ j \leq N$ where $i \neq j$, is the travel time, t_{ij}, from customer c_i to c_j and a distance traveled, $d(i, j) = d_{ij}$, that are symmetrical, i.e. $t_{ij} = t_{ji}$ and $d_{ij} = d_{ji}$. According to [30], the basic requirements of every customer, c_i, are as follows:

CR1: the quantity, q_i, of the product will be delivered by the vehicle, v_h.

CR2: The time, t_{ij}, required by the vehicle, V_h, moving from the depot, c_0, or a customer, c_i, to service the next customer, c_{i+1}, to unload the quantity, q_i, moving to the next customer or go back to the depot after servicing all the customers on that route or exhaust all the quantities carried.

CR3: The priority, δ, of the customer, c_i, to be serviced by the vehicle, V_h.

There is a set of identical vehicles, V. The carrying capacity of each vehicle, $v_h \in V$ is represented by Q_h.

Like the customers having some set-aside requirements, the vehicles also have the following requirements to be met as viewed by [2] thus:

VR1: the vehicle has a limited working duration, T_h, from the starting time, T_h^s, to the finishing time, T_h^f.

VR2: the fixed cost, FC_h, is the salaries/wages the drivers and the loaders/unloaders attached to the vehicles are paid including the vehicle maintenance.

VR3: the carrying capacity, Q_h, of the vehicle is the maximum load the vehicle can carry at a time.

For the authors in [30] to model the problem, the following general assumptions as it concerns the customers' requirements and vehicles' characteristics were itemized:

A1: The variable cost, VC_{ij}, is the least path cost traversed by the vehicle from customer c_i to the next customer c_j.

A2: The travel time, t_{ij}, is the corresponding duration of the vehicle spent from customer c_i to another customer c_j.

A3: The set $R_i = \{r_i(1), \dots, r_i(N)\}$ represents the routes for the vehicle V_h, where $r_i(N)$ represents the nth customer while N represents the number of customers on that route. Also, it is assumed that every route must terminate at the depot hence, $r_i(N+1) = 0$.

A4: The distance of the parking lot of the vehicle from which unloading is done to the warehouse or store of each customer is assumed to be equal. Hence, the time to unload per unit item is constant.

If the vehicle v_h serves the customer c_j immediately after servicing the customer c_i, then $\xi_{ijh} = 1$ otherwise, $\xi_{ijh} = 0$.

As opined by [2], [9], and [10], a typical routing problem with multiple priorities is considered to be a multi-objective problem. Where $Min\, J_1$ computes the least path or distance carrying cost, $Min\, J_2$ computes the fixed cost, $Max\, J_3$ is targeted at evaluating the priorities, and $Max\, J_4$ is set at calculating the OC which is the sum of the ERC and LRC. Hence,

$$Min\, J_1 = \alpha \sum_{i=1}^{N+x} \sum_{j=0}^{N+x} \left(\sum_{i=1}^{G} VC_i + \sum_{i=1}^{x} VC_G^i + \sum_{i=G+1}^{N-G} VC_i \right) \sum_{i=1}^{M} \xi_{ijh} \quad (35)$$

$$Min\, J_2 = \beta \sum_{h=1}^{M} \left(FC_h \sum_{j=1}^{N} \xi_{0jh} \right) \quad (36)$$

$$Max\, J_3 = \gamma \sum_{j=1}^{N}(\delta(c_j) \sum_{i=0}^{N}\sum_{k=1}^{M} \xi_{ijh}) \quad (37)$$

$$Max\, J_4 = \sum_{i=1}^{N+x}\sum_{j=0}^{N+x}(\sum_{i=1}^{G} c_i + \sum_{i=1}^{x} c_G^i + \sum_{i=G+1}^{N-G} c_i) \sum_{i=1}^{M} \xi_{ijh} \quad (38)$$

where α, β, and γ by [31] are arbitrary constants for weighting the terms (35), (36), and (37) corresponding to each objective.

The IVRP objective function with multiple priorities to which this work aimed at formulating is found combining all four objectives in (35), (36), (37), and (38) similar to [10] as:

$$Min\, J_1 + Min\, J_2 + Max\, J_3 + Max\, J_4 \quad (39)$$

$$= \alpha \sum_{i=1}^{N+x}\sum_{j=0}^{N+x}\left(\sum_{i=1}^{G} VC_i + \sum_{i=1}^{x} VC_G^i + \sum_{i=G+1}^{N-G} VC_i\right)\sum_{i=1}^{M}\xi_{ijh} + \beta \sum_{h=1}^{M}\left(FC_h \sum_{j=1}^{N}\xi_{0jh}\right) + \gamma \sum_{j=1}^{N}\left(\delta(c_j)\sum_{i=0}^{N}\sum_{h=1}^{M}\xi_{ijh}\right)$$

$$+ \sum_{i=1}^{N+x}\sum_{j=0}^{N+x}(\sum_{i=1}^{G} c_i + \sum_{i=1}^{x} c_G^i + \sum_{i=G+1}^{N-G} c_{G+i}) \sum_{i=1}^{M} \xi_{ijh} \quad (40)$$

Subject to:

$$\sum_{i=0}^{N}\sum_{h=1}^{M}\xi_{ijh} \le 1, \quad (41)$$

$$\sum_{i=0}^{N}\xi_{iph} - \sum_{j=0}^{N}\xi_{pjh} = 0,\ \ p = 0, \dots, N \quad (42)$$

$$\sum_{i=1}^{N}\left(q_i \sum_{j=0}^{N}\xi_{ijh}\right) \le Q_h,\ \ h = 1, \dots, \quad (43)$$

$$\sum_{i=0}^{N}\sum_{j=0}^{N} t_{ij}\,\xi_{ijh} + \sum_{i=1}^{N}\left(\sum_{j=0}^{N}\xi_{ijh}\right) \le T_h^f - T_h^s, \quad (44)$$

$$\sum_{j=1}^{M}\xi_{0jh} \le 1, \quad (45)$$

$$y_i - y_j + N\sum_{k=1}^{M}\xi_{ijh} \le N - 1,\ i \ne j \quad (46)$$

$$\xi_{ijh} \in \{0,1\}\ \forall\ i, j, h \quad (47)$$

The constraint in (41) expresses the fact that, at most, a customer can only be serviced once by a vehicle. Constraint (42) stresses the fact that any vehicle that visits a particular customer must ultimately depart from such a customer. Constraint (43) relates to the carrying capacity of the vehicle. Constraint (44) states the working time limitations on each route. Constraint (45) stresses the use of a vehicle at most once. The relation (46) with y_i arbitrary, is the sub-tour-elimination condition attached to the Travelling Salesman Problem (TSP) by [31] and as opined by [32] and [33] in VRP. The sub-tour elimination ensures that each vehicle routes through the depot. The constraint in (47) is the integrality conditions.

From the formulated IVRP objective above, should the IVRP aim to determine the priorities alone then, the series in (35), (36), and (38) are set as zero in (40). If the IVRP is aimed at determining the priorities as well as the costs then, the series in (38) is set as zero. Where the target is to calculate the intermittencies, (35), (36), and (37) are set at zero in (40) but, if the aim is to compute the variable cost, fixed cost, the priorities, and the intermittencies then, (40) holds.

However, the central idea behind the IVRP is to assist the dispatcher manager in planning the distribution/collection network ahead of time such that a customer gets the desired quantity and is delivered at the said time. It enables timely delivery, vehicle space, and capacity management of the vehicles. With these, it ensures that servicing of customers is based on the priorities such customers earlier set with a view to minimizing both the fixed and variable costs and maximizing the profit. With a proviso that, should an intermittent customer come in between the ERC, such customers' requests are also met without affecting the earlier planned routes, timing, and quantities.

VIII. CONCLUSION

Real-life situations that are hinged on incessant changes daily basis have made IVRP with multiple priorities unavoidable. As such, customers will not be lost to any close competitors, rather, more customers will be made hence, increasing the profit margin. The development and advancement in information technology have enormously contributed to solving this class of problems. The use of GSM, GPS, and network facilities has made become less difficult. Otherwise, it would have been a mirage and unattainable.

An anticipatory quantity must be carried along by the vehicle. This will give room for additional anticipatory time to cover the supply and delivery. There should be information interconnectivity between the depot and the customers via the vehicles in the supply chain.

While investigating IVRP with multiple priorities, randomly generated data were used against real-life data. Reasons for this are connected to: firstly, data randomly generated often enables an in-depth analysis. This is because the sets of data can be constructed such that other issues can be taken care of alongside. Secondly, most day-to-day activities involving IVRP with multiple priorities are needed for holistic analyses of the routing problem because all the data are not captured. The detailed information about the locations of all the vehicles is not known at the time the LRC request is received. This is one of the missing data items in real-life business activities hence, necessitates randomly generating such data.

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