On the Introduction of a Constructed Operator to an Extended Conjugate Gradient Method (ECGM) Algorithm

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Abstract**:** The development of a Conjugate Gradient Method (CGM) algorithm has immensely contributed to the solution of optimization problems due to its quadratic convergence property. Hinging on the CGM algorithm convergence property for optimization problems, this paper stresses the construction of a control operator that is introduced to a CGM algorithm that makes it amenable to solve optimal control problems. The introduction of the operator in CGM algorithm gave rise to an Extended Conjugate Gradient Method (ECGM) algorithm adopted for solving Continuous Time Regulator Problems (CLRP) that is constrained by delay differential equation. Unlike the similar control operators constructed in time past designated to solve either the Mayer or the Lagrange cost forms of the CLRP, the adoption of this control operator in ECGM algorithm will effectively and robustly takes care of the Mayer form, the Lagrange cost forms, and the Bolza cost form of the CLRP. The resulting algorithm on the introduction of the control operator to ECGM was tried on a number CLRP exhibiting an improved convergence profile over the classical methods hence widening the range of problems to which the ECGM algorithm can be employed to solve.

Index Terms: Delay Differential State Equation, Extended Conjugate Gradient Method, Linear Operator, Optimal Control, Regulator Problem.

I. INTRODUCTION

In the last three decades, researchers have intensified efforts in constructing different control operators that are introduced to Conjugate Gradient Method (CGM) algorithm that makes is employable to solve control problems that have the Mayer form and the Lagrange form objective function with or without delay differential equation in the constraint.

The development and the application of the linear operator serves as the pivot to this paper. Though, the construction and development of similar operators are not relatively new. In time past, the authors in [1], [2], and [3] have constructed similar control operators specifically for Lagrange form of Continuous Time Linear Regulator Problem (CLRP). All such constructions were without the delay parameter in the constraint as:

$$Min_{(x,u)} \int_{t_0}^{t_f} \{ax^2(t) + \ bu^2(t)\}dt \quad (1)$$

Subject to

$$\dot{x}(t) = cx(t) + du(t), \quad t_0 \le t \le t_f, \quad (2)$$

$$x(t_0) = x_0; \quad (3)$$

where a, b, c, and d are given constants with $a > 0, b > 0$; x_0 and t_f are specified values, $\dot{x}(t)$ represents the state variable derivative, $x(.)$ with respect to time, t.

Another closed such operator constructed was that of the control vector represented by $u(.)$ and the Lagrange form of CLRP with delay parameter in the constraint as:

$$Min_{(x,u)} \int_{t_0}^{t_f} \{Ax^2(t) + Bu^2(t)\}dt \quad (4)$$

Subject to

$$\dot{x}(t) = C_1x(t) + C_2x(t-r) + Du(t), \quad t_0 \le t \le t_f, \quad (5)$$

$$x(t) = h(t); \quad -r \le t \le 0 \quad (6)$$

where $A, B > 0$; the delay parameter $r > 0$ and t_f are given. The given constants C_1, C_2, and D are not necessarily positive.

Sequel to the constructions of similar operators and solution to CLRP, the authors in [4] developed a variant to solve the energized equation. Of all the afore mentioned similar constructions, none treated the Bolza form of the CLRP. The Bolza form of CLRP subject to delay differential equation have been left unattended to over years hence, this work is geared towards bridging the gaps and providing an algorithm that will holistically capture both the Mayer, the Lagrange, and the Bolza forms of CLRP.

The typical Bolza form performance index viewed by [5] and [6] is to minimize the Continuous Time Linear Regulator Problem opined by [7] as:

$$J(x, t_0, t_f, u(.)) = \tfrac{1}{2}x^T(t_f)Hx(t_f) + \tfrac{1}{2}\int_{t_0}^{t_f}\{x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\}dt \quad (7)$$

Subject to the delay differential state equation

$\dot{x}(t) = C_1 x(t) + C_2 x(t-r) + Du(t), t_0 \leq t \leq t_f,$ (8)

$x(t) = h(t)$, with $-r \leq t \leq 0$ (9)

where the $n \times n$ matrices H, $Q(t) \in \mathcal{R}$ are symmetric and positive semi-definite. The $m \times m$ matrix $R(t) \in \mathcal{R}$ is symmetric and positive definite. Both the starting and the finishing times, t_0 and t_f, respectively are specified. The n-dimensional state vector $x(t)$ and the m-dimensional input control vectors $u(t)$ are not constrained by any boundary. The following constants $C_1, C_2,$ and D are given but are not necessarily positive. The delay variable, $r > 0$ and $h(t)$ are continuously piecewise function whose order are defined exponentially on $[-r, 0]$. However, if the value of $H = 0$ then, (7) is reduced to the Lagrange cost form of the CLRP. Consequently, (7) will be reduced to the Mayer cost form of the CLRP if $Q(t)$ and $R(t)$ are both zero matrices.

In the opinions of [2] and [8], the delay differential constraint in (8) constitutes and comprises of essential model used widely by many researchers. To fast track the operator construction, the CLRP Bolza form performance measure depicted in (7) has to be recast into the CLRP Lagrange cost form thus:

$$J = \int_{t_0}^{t_f} \{\frac{d}{dt}(\frac{1}{2}x^T(t_f)Hx(t_f))\}\,dt + \frac{1}{2}\int_{t_0}^{t_f}\{x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\}dt \quad (10)$$

$$J = \int_{t_0}^{t_f}\{x^T(t)H\dot{x}(t)\}\,dt + \frac{1}{2}\int_{t_0}^{t_f}\{x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\}dt \quad (11)$$

$$J = \int_{t_0}^{t_f}\left\{x^T(t)H\dot{x}(t) + \frac{1}{2}x^T(t)Q(t)x(t) + \frac{1}{2}u^T(t)R(t)u(t)\right\}dt. \quad (12)$$

Customarily with the penalty function schemes (12) that is constrained by (8), it may be compressed in the unconstrained equivalent form:

$$\langle z(t), \boldsymbol{G}z(t)\rangle_k = \min_{(x,u)} \int_{t_0}^{t_f}\{x^T(t)H\dot{x}(t) + \frac{1}{2}x^T(t)Q(t)x(t) + \frac{1}{2}u^T(t)R(t)u(t) + \mu\|C_1x(t) + C_2x(t-r) + Du(t) - \dot{x}(t)\|^2\}dt \quad (13)$$

$x(t) = h(t)$; with $t \in [-r, 0]$ (14)

where $\mu > 0$ is the penalty parameter and $\|C_1x(t) + C_2x(t-r) + Du(t) - \dot{x}(t)\|^2$ is the penalty term.

Let the product space $\tilde{k}$ be represented by:

$$\tilde{k} = \mathcal{H}[t_0, t_f] \times \ell_2[t_0, t_f] \times \ell_2[-r, 0] \quad (15)$$

as the Sobolev space and the strictly continuous function $\mathcal{H}[t_0, t_f]$, whereby both $x(.)$ and $\dot{x}(.)$ are integrable squares within the closed intervals $[t_0, t_f]$. Also, let the Hilbert space, $\ell_2[\alpha, \beta]$, be a real valued equivalent set of functions on $[\alpha, \beta]$ with the norm represented by:

$$\|f(.)\|_{\ell_2[\alpha,\beta]} = \left(\int_\alpha^\beta |f(t)|^2 dt\right)^{\frac{1}{2}}, f(.) \in \ell_2[\alpha, \beta]. \quad (16)$$

Hence, the inner product $\langle .,. \rangle_{\tilde{k}}$ on $\tilde{k}$ is represented by:

$$\langle .,. \rangle_{\tilde{k}} = \langle .,. \rangle_{\mathcal{H}[t_0,t_f]} + \langle .,. \rangle_{\ell_2[t_0,t_f]} + \langle .,. \rangle_{\ell_2[-r,0]}. \quad (17)$$

Suppose $z(t) \in \tilde{k}$ denotes the ordered triple pair:

$$z^T(t) = (x(t), u(t), h(t));\ x(t) \in \mathcal{H}[t_0, t_f],\ u(t) \in \ell_2[t_0, t_f], \text{ and } h(t) \in \ell_2[-r, 0] \quad (18)$$

then, on $\tilde{k}$ one seeks to determine the operator $\boldsymbol{G}$ such that when $\tilde{k}$ is an appropriately chosen Hilbert space (13) will hold. While from (15), it follows that, $z(t) \in \tilde{k}$ has the norm

$$\|z(t)\|_{\tilde{k}}^2 = \|x(t)\|_{\mathcal{H}[t_0,t_f]}^2 + \|u(t)\|_{\ell_2[t_0,t_f]}^2 + \|h(t)\|_{\ell_2[-r,0]}^2. \quad (19)$$

For notational convenience, simply $z(t)$ will be written as $z^T(t) = (x(t), u(t), h(t))$ for the ordered triplet while keeping in mind that the domain of definition of $h(t)$ is the closed interval $[-r, 0]$, while both $x(t)$ and $u(t)$ are defined on the interval $[t_0, t_f]$.

The operator, G**,** is then introduced into the CGM algorithm to make it amenable to solve control problems as in (13). For the completeness sake, a recap of the CGM algorithm follows next.

II. CONJUGATE GRADIENT METHOD (CGM) ALGORITHM

Considering the descent conjugate with a functional, $f(x)$ on the Hilbert space $\mathcal{H}$ where $f(x)$ is a Taylor series expansion truncated after the second ordered terms as follows:

$$f(x) = f_0 + \langle a, x\rangle + \frac{1}{2}\langle x, Gx\rangle \quad (20)$$

where $\langle . \rangle$, is the scalar product of $\mathcal{H}$. The linear operator G is presumed symmetric and positive definite so that the functional $f(x)$ has a definite minimum x^* in the space $\mathcal{H}$. Iteratively, the CGM algorithm that is employed for determining the minimum x^* of $f(x)$ in $\mathcal{H}$ by [9] is outlined thus:

Step 1: The starting member $x_0 \in \mathcal{H}$ is a guessed value while the other elements of the sequence will be computed using the relations outlined in steps 2 through step 6 of the algorithm.

Step 2: Compute the direction of the descent search

$$p_0 = -g_0 \quad (21a)$$

Step 3: Set $x_{i+1} = x_i + \alpha_i p_i$; where $\alpha_i = \frac{\langle g_i, g_i\rangle_{\mathcal{H}}}{\langle p_i, Gp_i\rangle_{\mathcal{H}}}$ (21b)

Step 4: Compute $g_{i+1} = g_i + \alpha_i G p_i$ (21c)

Step 5: Set $p_{i+1} = -g_{i+1} + \beta_i p_i$, $\beta_i = \frac{\langle g_{i+1}, g_{i+1}\rangle_{\mathcal{H}}}{\langle g_i, g_i\rangle_{\mathcal{H}}}$ (21d)

Step 6: Test for the convergence. If for some i, $g_i = 0$ then, truncate the process. Otherwise, choose $i = i + 1$ and return to step 3.

At the i-th step of the iterative procedure of the CGM algorithm from steps 2 through 6 the, p_i denotes the descent search direction, α_i stands for the step length of the sequence $\{x_i\}$, and at g_i stands for the slope of F. The crucial role played by the operator G is revealed in steps 3, 4, and 5 of the algorithm to calculate the step size of the descent sequence and to compute the direction of conjugate search. The algorithm usability hinges on the available information from the linear operator G.

More often, for discrete optimization problems the linear operator G is readily determined as contained in [9] and this class of problem enjoys the computational beauty of the CGM as a scheme since the algorithm has quadratic converges while a little more calculation will be required in each iteration. It is in the light of this that the CGM properties make it an interesting computational scheme with a strong appeal for its digital computer implementation as viewed by [2] and [10].

Howbeit, for the class of problem mentioned in this work, the CGM algorithm so presented could be deterred because, an equivalent to the linear operator G that will satisfy (13) is not readily found. Also, $h(s)$ in (14) is a specified piecewise continuous function that is of an exponential order on $[-r, 0]$.

Then, for $\mu > 0$ there exists a control operator G, with $G: \widetilde{k} \times \widetilde{k} \to \mathcal{R}$ such that:

$$\langle z(t), Gz(t) \rangle_{\widetilde{k}} = \int_{t_0}^{t_f}\{Ax^2(t) + Bu^2(t) + \mu\|\dot{x}(t) - C_1x(t) - C_2x(t-r) - Du(t)\|^2\}dt \quad (22)$$

where $z(t) = (x(t), u(t), h(t))^T$. Furthermore, G is given by:

$$(Gz(t)) \equiv \begin{pmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{pmatrix}\begin{pmatrix} x(t) \\ u(t) \\ h(t) \end{pmatrix} = \begin{pmatrix} (G_{11}x)(t) + (G_{12}u)(t) + (G_{13}h)(t) \\ (G_{21}x)(t) + (G_{22}u)(t) + (G_{23}h)(t) \\ (G_{31}x)(t) + (G_{32}u)(t) + (G_{33}h)(t) \end{pmatrix} \quad (23)$$

where the composite operators of G_{ij}, for $1 \le i \le 3$ and $1 \le j \le 3$ are specified explicitly in [2].

III. Main Result

The following theorem herein contains the results for Problem 1.

Theorem 1: Let $\widetilde{K}$ be the product space given as:

$$\widetilde{K} = W^{1,2}[t_0, t_f] \times \ell^2[t_0, t_f] \times \ell^2[-r, 0] \quad (24)$$

of the Sobolev space of a function $x(.)$ that is absolutely continuous on closed interval $[t_0, t_f]$ and the regular real-valued Hilbert space $\ell^2[t_0, t_f]$ of a Lebesgue measurable functions that are square integrable on the interval $[t_0, t_f]$. Then, for $\mu > 0$ there exists a usual control operator, G, with $G: \widetilde{K} \times \widetilde{K} \to \mathcal{R}$ such as in (13) where the constituents of the operators (23) are given as:

$$(G_{11}x(t)) = -\mu\dot{x}(0)Sinh(t_f) + \int_0^{t_f} \mu\dot{x}(s)Sinh(t_f - s)ds - \int_0^{t_f} Sinh(t_f - s)\left[\left(\tfrac{1}{2}Q + \mu C_1^2 + C_2^2\right)x(s) + (H - 2\mu C_1)\dot{x}(s) + 2\mu C_1C_2x(s+r) - 2\mu C_2\dot{x}(s+r)\right]ds + \left[\left(\tfrac{1}{2}Q + \mu C_1^2 + C_2^2\right)x(0) + (H - 2\mu C_1)\dot{x}(0) + 2\mu C_1C_2x(r) - 2\mu C_2\dot{x}(s+r)\right]Cosh(t) + \frac{Sinht}{Sinht_f}\{\left(\tfrac{1}{2}Q + \mu C_1^2 + C_2^2\right)x(t_f) + (H - 2\mu C_1)\dot{x}(t_f) + 2\mu C_1C_2x(t_f - r) - 2\mu C_2\dot{x}(t_f - r) - \left[\left(\tfrac{1}{2}Q + \mu C_1^2 + C_2^2\right)x(0) + (H - 2\mu C_1)\dot{x}(0) + 2\mu C_1C_2x(r) - 2\mu C_2\dot{x}(r)\right] \times Cosh(t_f) + \int_0^{t_f} Sinh(t_f - s)[\left(\tfrac{1}{2}Q + \mu C_1^2 + C_2^2\right)x(s) + (H - 2\mu C_1)\dot{x}(s) + 2\mu C_1C_2x(s+r) - 2\mu C_2\dot{x}(s+r)]ds + \mu\dot{x}(0)Sinh(t_f) - \int_0^{t_f} \mu\dot{x}(s)Cosh(t_f - s)ds\}\,;\ 0 \le t \le t_f \quad (25)$$

$$(G_{21}\,x(t)) = -\,2\mu D\dot{x}(t);\ \ 0 \le t \le t_f \quad (26)$$

$$(G_{31}x(t)) = 2\mu C_1C_2x(t+r) - 2\mu C_2\dot{x}(t+r) + \mu C_2^2x(t)\,;\ -r \le t \le 0 \quad (27)$$

$$(G_{12}u(t)) = 2\mu Du(0)Sinh(t_f) + \int_0^{t_f} 2\mu Du(s)Cosh(t_f - s)ds - \int_0^{t_f} Sinh(t_f - s)[2\mu C_1u(s) + 2\mu C_2Du(s+r)]ds + [2\mu C_1Du(0) + 2\mu C_2Du(r)]Cosh(t) + \frac{Sinh(t)}{Sinh(t_f)}\{2\mu C_1Du(t_f) + 2\mu C_2Du(t_f - r) - 2\mu C_1Du(0) - 2\mu C_2Du(r) + 2\mu Du(o)Sinh(t_f) - \int_0^{t_f} 2\mu Du(s)Cosh(t_f - s)ds + \int_0^{t_f - r} Sinh(t_f - r - s)[2\mu C_1Du(s) + 2\mu C_2Du(s+r)]ds\ ;\ 0 \le t \le t_f \quad (28)$$

$$(G_{22}\,u(t)) = \tfrac{1}{2}Ru(t) + \mu D^2u(t)\ \ ;\ \ 0 \le t \le t_f \quad (29)$$

$$(G_{32}\,u(t)) = 2\mu C_2Du(t+r)\ \ \ ;\ -r \le t \le 0 \quad (30)$$

$$(G_{13}h(t)) = 2\mu C_2h(0)Sinh(r) - \int_0^r Cosh(r-\tau)[2\mu C_2h(0)]d\tau - \int_0^r Sinh(r-\tau)[2\mu C_1C_2h(\tau)]d\tau + \frac{Sinh(t)}{Sinh(t_f)}\{2\mu C_1C_2h(r) - 2\mu C_1C_2h(0)Cosh(r) - 2\mu C_2h(0)Sinh(r) + \int_0^r Cosh(r-\tau)[2\mu C_2h(\tau)]d\tau + \int_0^r Sinh(r-\tau)[2\mu C_1C_2h(r)]d\tau\} + 2\mu C_1C_2h(0)Cosh(t)\,;\ 0 \le t \le r \quad (31a)$$

$$(G_{13}\,h(t)) = 0\,;\ r = 0 \quad (31b)$$

$$(G_{23}\,h(t)) = \begin{cases} 2\mu C_2Dh(s); & 0 \le s \le r \\ 0; & r = 0 \end{cases} \quad (32)$$

$$(G_{33}\,h(t)) = \begin{cases} \mu C_2h(t); & 0 \le s \le r \\ 0; & r = 0 \end{cases} \quad (33)$$

For the completeness of the proof of this theorem, the following fundamentals are necessary.

Definition 1: Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces (over $\mathcal{R}$). A bilinear form (functional) Q on $\mathcal{H}_1 \times \mathcal{H}_2$ is a mapping $Q: \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{R}$ such that for all $x, x_1, x_2 \in \mathcal{H}_1$, $y, y_1, y_2 \in \mathcal{H}_2$ and $\alpha, \beta \in \mathcal{R}$ the following properties hold:

$$Q(x_1 + x_2, y) = Q(x_1, y) + Q(x_2, y); \quad (34)$$

$$Q(x, y_1 + y_2) = Q(x, y_1) + Q(x, y_2); \quad (35)$$

$$Q(\alpha x, y) = \alpha Q(x, y); \quad (36)$$

$$Q(x, \beta y) = \beta Q(y, x); \quad (37)$$

If in definition 1, it is supposed that $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ and in addition to satisfying (34) through (37) on $\mathcal{H}$ then Q will also satisfy

$$Q(x, y) = Q(y, x) \quad (38)$$

then, Q is said to be a bilinear, Hermitian (or self-adjoin) form on $\mathcal{H}$.

Next, recall the following theorems due to F, Reisz:

Theorem 2: Every bounded linear functional T on a Hilbert space $\mathcal{H}$ can be denoted in terms of the inner product, namely; $T(x) = \langle x, z\rangle_{\mathcal{H}}, x \in \mathcal{H}$

where z is uniquely determined by T. ■

Theorem 3: Let $\mathcal{H}_1$and $\mathcal{H}_2$ be Hilbert spaces and $Q: \mathcal{H}_1 \times \mathcal{H}_2 \rightarrow \mathcal{R}$ a bounded bilinear form. If $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$ then Q has a representation that

$$Q(x, y) = \langle Sx, y\rangle_{\mathcal{H}_2}$$

where $S: \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is a distinctively computed closed linear operator. Now, let the Hilbert space $\mathcal{H}$ of the continuous linear operator be represented by $T: \mathcal{H} \rightarrow \mathcal{H}$ and the fixed $y \in \mathcal{H}$ and $x \in \mathcal{H}$ then, defines

$$f_y(x) = \langle Tx, y\rangle_{\mathcal{H}} \quad (39)$$

where f_y is a continuous linear functional on $\mathcal{H}$ and the following remarks become necessary. ■

Remark:

(i) From Theorem 2, there exists a unique element $y^* \in \mathcal{H}$ such that $f_y(x) = \langle x, y^*\rangle_{\mathcal{H}}$.

(ii) If under the hypothesis of Theorem 3 choose $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ then it follows that every bounded, self-adjoint linear operator T on $\mathcal{H}$ generates a bounded, bilinear, and Hermitian form $T\langle x, y\rangle_{\mathcal{H}} = \langle Tx, y\rangle_{\mathcal{H}} = \langle x, Ty\rangle_{\mathcal{H}}$ on $\mathcal{H}$ and G vice-versa. Theorems 2 and 3 together with the Remark 1, provide the framework for the construction of the control operator, $\boldsymbol{G}$.

Definition 2: Let $f(t)$ and $g(t)$ be sectionals continuous functions on some domain [a, b]. The convolution, $f * g(t)$ of f and g is defined by the expression

$$f * g(t) = \int_0^t f(u) g(t - \tau) du. \quad (40)$$

Theorem 4: Let $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$ be the Laplace transforms of f and g respectively. Then, by [11], the transforms $\mathcal{L}\{f * g(t)\}$ of the convolution of f and g is the product of the Laplace transforms of f and g, that is, $\mathcal{L}\{f * G(t)\} = F(s)G(s) = \int_0^t f(\tau) g(t - \tau) d\tau$. (41)

Proof: By the definition of $G(s)$ and the second shifting theorem, for each fixed τ $(\tau \geq 0)$, gives

$$e^{-s\tau} G(s) = \mathcal{L}\{ g(t - \tau) u(t - \tau)\}$$

$$= \int_0^\infty e^{-st} g(t - \tau) u(t - \tau) dt$$

$$= \int_\tau^\infty e^{-st} g(t - \tau) dt \quad (42)$$

where $s > k$. From this and the definition of $F(s)$ obtains

$$F(s)G(s) = \int_0^\infty e^{-s\tau} f(\tau) G(s) d\tau$$

$$= \int_0^\infty f(\tau) \int_0^\infty e^{-s\tau} g(t - \tau) dt d\tau \quad (43)$$

where $s > k$. Here, an integration over τ to ∞ and then over τ from 0 to ∞ is observed. Considering the constraint (8) of (7) involving a delay term. Therefore, the Laplace transform of the delay function required in the following Lemma are provided.

Lemma 1: Let $\tau > 0$ be given, and let $\mathcal{L}\{f(t)\} = F(s)$ denotes the Laplace transform of a function $f(t)$ such that $f(t) = h(t)$ for $-\tau \leq t \leq 0$. Then, the Laplace transform of $f(t - \tau)$ is given by:

$$\mathcal{L}\{f(t - \tau)\} = e^{-s\tau} g(s) + e^{-s\tau} F(s)$$

where $g(s) = \int_{-\tau}^0 e^{-st} h(t) dt$.

Proof:

$$\mathcal{L}\{f(t - \tau)\} = \int_0^\infty e^{-st} f(t - \tau) dt = \int_{-\tau}^\infty e^{-s(t+\tau)} f(t) dt$$

$$= e^{-s\tau}[\int_{-\tau}^0 e^{-st} f(t) dt + \int_0^\infty e^{-st} f(t) dt] = e^{-s\tau}[\int_{-\tau}^0 e^{-st} h(t) dt] + e^{-s\tau} F(s)$$

On setting $g(s) = \int_{-\tau}^0 e^{-st} h(t) dt$, the proof of the lemma is concluded. ■

Finally, according to [12], invoking the next lemma which plays a key role in the calculus of variations, in the construction of the control operation, $\boldsymbol{G}$. First, define the space $G_n(0, T)$.

Definition 3: $G_n(0, T)$ is the space of all continuous functions $y(t)$, in the interval $0 \leq t \leq T$ that are continuously differentiable up to the n-times on $[0, T]$ with the norm $\|y\|_n$ provided by

$$\|y\|_n = \sum_{i=1}^{n} \max_{0\le t\le T}\left|y^{(i)}(t)\right|$$
$$= \max_{0\le t\le T}\left|y^{(1)}(t)\right| + max_{0\le t\le T}\left|y^{(2)}(t)\right|$$
$$+max_{0\le t\le T}\left|y^{(3)}(t)\right| + \cdots + max_{0\le t\le T}\left|y^{(n)}(t)\right|$$

where $y^{(i)}(t), (1 \le i \le n)$ presents the *i-th* derivative of $y(t)$ and *n* is a fixed integer.

Lemma 2: Let $n \ge 0$ be an integer and suppose $\mathcal{R}^n \in [0,T]$ stands for the space of all real-valued functions $y(t), 0 \le t \le T$ which are continuously differentiable n-times on $[0,T] \subset \mathcal{R}$ with the norm $\|y\|_n$ given by:

$$\|y\|_n = \sum_{i=0}^{n} max_{0\le t\le T}\left|y^{(i)}(t)\right| \quad (44)$$

where the ith derivative of $y(t)$ is being represented by $\left|y^{(i)}(t)\right|$.

Then, if $\alpha(t)$ and $\beta(t)$ are continuous in $[a,b]$ and if $\int_a^b[\alpha(t)y(t) + \beta(t)\dot{y}(t)]dt = 0$ for every function $g(t) \in \mathcal{R}^1(a,b)$ such that $g(a) = g(b) = 0$, then $\beta(t)$ is differentiable and $\frac{d}{dt}\big(\beta(t)\big) = \alpha(t)$ for all $t \in [a,b]$.

Proof of Theorem 1:

It is pertinent to recall the unconstrained equivalent form (13) and for succeeding development, it will connect with (13) the functional, $R_\mu(z_1,\ z_2)$ stated as:

$$R_\mu(z_1,z_2) = \int_0^{t_f}\{\tfrac{1}{2}x_1(t)Qx_2(t) + \tfrac{1}{2}u_1(t)Ru_2(t) + x_1(t)H\,\dot{x}_2(t) + \mu C_1^2x_1(t)x_2(t) + \mu C_1C_2x_1(t)x_2(t-r) + \mu C_1Dx_1(t)u_2(t) - \mu C_1x_1(t)\dot{x}_2(t) + \mu C_1C_2x_1(t)x_2(t-r) + \mu C_2^2x_1(t-r)x_2(t-r) + \mu C_2Dx_1(t-r)u_2(t) - \mu C_2x_1(t-r)\dot{x}_2(t) + \mu C_1Dx_1(t)u_2(t) + \mu C_2Dx_1(t-r)u_2(t) + \mu D^2u_1(t)u_2(t) - \mu Du_1(t)\dot{x}_2(t) - \mu C_1\,x_1(t)\dot{x}_2(t) - \mu C_2x_1(t-r)\dot{x}_2(t) - \mu Du_1(t)\dot{x}_2(t) + \mu\dot{x}_1(t)\dot{x}_2(t)\}dt \quad (45)$$

where $z_1^T = \big(x_1(t),\ u_1(t),\ h_1(t)\big), z_2^T = \big(x_2(t),\ u_2(t),\ h_2(t)\big)$ belonging to space $\tilde{k}$ that are stated in (24) are the triple ordered pair. Under the equivalent relationships, it follows that the form in (35) is equivalent to that in (38) as:

$$\begin{Bmatrix} x_1(t) \equiv x_2\,(t) = x(t), & 0 \ge t \ge t_f, \\ \dot{x}_1(t) \equiv \dot{x}_2(t) = \dot{x}(t), & 0 \ge t \ge t_f, \\ u_1(t) \equiv u_2\,(t) = u(t), & 0 \ge t \ge t_f, \\ h_1(t) \equiv h_2\,(t) = h(t), & -r\ \ge t \ge 0. \end{Bmatrix} \quad (46)$$

For proof, see [9]. The following are vital to our subsequent developments:

Proposition 1:

Prove that $R_\mu(z_1,z_2)$ a self-adjoin form on $\tilde{k}$ is bounded and bilinear.

Proof:

By bi-linearity and self-adjointness of $R_\mu(z_1,z_2)$, it's obvious that, from the definition and its boundedness, it follows that $z_i(t) = \big(x_i(t),\ u_i(t),\ h_i(t)\big)^T, i = 1,2$ is bounded.

Remark 2: By reasons of Proposition 1 and consequential on the representation theorem of Reiess on the Hilbert spaces in [13], it implies that $R_\mu(z_1,z_2)$ induces the exclusively determined and closed linear operator G on $\tilde{k}$ with the delineation:

$$R_\mu(z_1,z_2) = \langle Gz_1,z_2\rangle_{\tilde{k}} = \langle z_1,Gz_2\rangle_{\tilde{k}} = R_\mu(z_1,z_2) \quad (47)$$

which makes it is obvious that G is a self-adjoint on $\tilde{k}$ as long as $R_\mu(z_1,z_2)$ does.

Linearly related to the delay term $x(t-r)$ is the prescribed initial function $h(t)$ in the sense that, for $t \in [0,t_f]$:

$$x(t-r) = x(s) = \begin{Bmatrix} h(s); & s \in [-r,0] \\ x(s); & s \in [0,t_f-r \end{Bmatrix}, h(0) = x(0). \quad (48)$$

It follows from that, when $h(t) \equiv 0$, then, $x(t-r) = x(t)$.

Now consider the equivalence:

$$\langle z_1,Gz_2\rangle = R\mu\langle z_1,z_2\rangle \quad (49)$$

This is rather convenient for the developments and on setting $h_2(t) \equiv u_2(t) = 0$ in (23) gives:

$$(\mathrm{G}z_2(t)) \equiv \begin{pmatrix}(G_{11}x_2)(t)\\(G_{21}x_2)(t)\\(G_{31}x_2)(t)\end{pmatrix}\begin{pmatrix}y_{11}(t)\\y_{21}(t)\\y_{31}(t)\end{pmatrix} \quad (50)$$

where the functions $y_{11}(t), y_{21}(t),$ and $y_{31}(t)$ must be determined in order to obtain the following $(G_{11}x_2)(t), (G_{21}x_2)(t),$ and $(G_{31}x_2)(t)$. By equivalencies of (47) and when $h_2(t) = 0$, it implies that $x_2(t-r) = x_2(t)$ from (45) that led to obtain the functional:

$$\langle z_1(t),z_2(t)\rangle = \int_0^{t_f}\{\tfrac{1}{2}x_1(t)Qx_2(t) + x_1(t)H\,\dot{x}_2(t) + \mu C_1^2x_1(t)x_2(t) + \mu C_1C_2x_1(t)x_2(t-r) - \mu C_1x_1(t)\dot{x}_2(t) + \mu C_1C_2x_1(t)x_2(t-r) + \mu C_2^2x_1(t-r)x_2(t-r) - \mu C_2x_1(t-r)\dot{x}_2(t) - \mu Du_1(t)\dot{x}_2(t) - \mu C_1\,x_1(t)\dot{x}_2(t) - \mu C_2x_1(t-r)\dot{x}_2(t) - \mu Du_1(t)\dot{x}_2(t) + \mu\dot{x}_1(t)\dot{x}_2(t)\}dt \quad (51)$$

On simplifying (51) further leads to:

$$\langle z_1(t),z_2(t)\rangle = \int_0^{t_f}\{x_1(t)\left[\left(\tfrac{1}{2}Q + \mu C_1^2\right)x_2(t) + (H - 2\mu C_1)\dot{x}_2(t) + 2\mu C_1C_2x_2(t-r)\right] + x_1(t-r)[-2\mu C_2\dot{x}_2 + \mu C_2^2x_2(t-r)] + [u_1(t)(-2\mu D) + \mu\dot{x}_1(t)]\dot{x}_2(t)\}dt \quad (52)$$

And on further simplification of (52) gives rise to:

$$\langle z_1(t),z_2(t)\rangle = \int_0^{t_f}\{x_1(t)\,[\left(\tfrac{1}{2}Q + \mu C_1^2\right)x_2(t) + (H - 2\mu C_1)\dot{x}_2(t)] + x_1(t-r)[2\mu C_1C_2x_2(t) - 2\mu C_2\dot{x}_2(t) + \mu C_2^2x_2(t-r)] + [u_1(t)(-2\mu D) + \mu\dot{x}_1(t)]\dot{x}_2(t)\}dt \quad (53)$$

On the introduction of (48) to the integrand in (53) leads to:

$$\int_0^{t_f}\{x_1(t-r)[2\mu C_1C_2x_2(t) - 2\mu C_2\dot{x}_2(t) + \mu C_2^2x_2(t-r)]\}dt. \quad (54)$$

Replacing $s + r = t$ and $-r = 0$ in (54) gives rise to:

$$= \int_{-r}^{t_f-r}\{x_1(s)[2\mu C_1C_2x_2(s+r) - 2\mu C_2\dot{x}_2(s+r) + \mu C_2^2x_2(s)]\}ds \quad (55)$$

$$= \int_{-r}^{0}\{x_1(s)[2\mu C_1C_2x_2(s+r) - 2\mu C_2\dot{x}_2(s+r) + \mu C_2^2x_2(s)]\}ds + \int_0^{t_f-r}\{x_1(s)[2\mu C_1C_2x_2(s+r) - 2\mu C_2\dot{x}_2(s+r) + \mu C_2^2x_2(s)]\}ds$$

$$= \int_{-r}^{0}\{h_1(t)[2\mu C_1C_2x_2(t+r) - 2\mu C_2\dot{x}_2(t+r) + \mu C_2^2x_2(t)]\}dt + \int_0^{t_f-r}\{x_1(t)[2\mu C_1C_2x_2(t+r) - 2\mu C_2\dot{x}_2(t+r) + \mu C_2^2x_2(t)]\}dt. \quad (56)$$

Since $x_1(t) = h_1(t)$ for $t \in [-r, 0]$, (46) holds. Also, for $t \in [0,\ t_f - r]$,

$$w_2(t) = \begin{cases} x_2(t+r); 0 \le t \le t_f - r \\ 0\ ; \qquad t_f - r \le t \le t_f \end{cases} \quad (57)$$

Then, keeping track of the domain function definition $x_2(t+r)$ and the functional $\langle z_1(t), z_2(t)\rangle$ in (53), it may be expressed as follows:

$$\langle z_1(t), z_2(t)\rangle = \int_0^{t_f}\{x_1(t)\left[\left(\tfrac{1}{2}Q + \mu C_1^2\right)x_2(t) + (H - 2\mu C_1)\dot{x}_2(t)\right] + h_1(t)[2\mu C_1C_2x_2(t+r) - 2\mu C_2\dot{x}_2(t+r) + \mu C_2^2x_2(t)] + x_1(t)[2\mu C_1C_2x_2(t+r) - 2\mu C_2\dot{x}_2(t+r) + \mu C_2^2x_2(t)] + [u_1(t)(-2\mu D) + \mu\dot{x}_1(t)]\dot{x}_2(t)\}dt \quad (58)$$

Introducing (57) into (58), we get the expression:

$$\langle z_1(t), z_2(t)\rangle = \int_0^{t_f}\begin{Bmatrix} x_1(t)[\left(\tfrac{1}{2}Q + \mu C_1^2\right)x_2(t) + (H - 2\mu C_1)\dot{x}_2(t) + 2\mu C_1C_2w_2(t) - 2\mu C_2\dot{w}_2(t) \\ +\mu C_2^2x_2(t)]\ + h_1(t)[2\mu C_1C_2x_2(t+r) - 2\mu C_2\dot{x}_2(t+r) + \mu C_2^2x_2(t)] \\ +[u_1(t)(-2\mu D) + \mu\dot{x}_1(t)]\dot{x}_2(t) \end{Bmatrix}dt \quad (59)$$

$$\langle z_1(t), z_2(t)\rangle = \int_0^{t_f}\begin{Bmatrix} x_1(t)[\left(\tfrac{1}{2}Q + \mu C_1^2\right)x_2(t) + (H - 2\mu C_1)\dot{x}_2(t) + 2\mu C_1C_2w_2(t) - 2\mu C_2\dot{w}_2(t) \\ +\mu C_2^2x_2(t)]\ + h_1(t)[2\mu C_1C_2x_2(t+r) - 2\mu C_2\dot{x}_2(t+r) + \mu C_2^2x_2(t)] \\ +u_1(t)(-2\mu D)\dot{x}_2(t) + \mu\dot{x}_1(t)\dot{x}_2(t) \end{Bmatrix}dt \quad (60)$$

$$\langle z_1(t), z_2(t)\rangle = \int_0^{t_f}\{x_1(t)y_{11}(t) + \dot{x}_1(t)\dot{y}_{11}(t) + u_1(t)y_{21}(t) + h_1(t)y_{31}(t)\}dt \quad (61)$$

The quantities $y_{11}(t), y_{21}(t)$, and $y_{31}(t)$ which satisfy (60) have to be determined as in [14]. Towards this, let represent

$$\alpha(t) = \left(\tfrac{1}{2}Q + \mu C_1^2\right)x_2(t) + (H - 2\mu C_1)\dot{x}_2(t) + 2\mu C_1C_2w_2(t) - 2\mu C_2\dot{w}_2(t) \quad (62)$$

$$\beta(t) = \mu\dot{x}_2(t) \quad (63)$$

then, $\alpha(t) - y_{11}(t)$ and $\beta(t) - \dot{y}_{11}(t)$ are continuous functions on $[0, t_f]$ and for $x_1(.) \in G_1[0, t_f]$ such that $x_1(0) = 0 = x_1(t_f)$, (60) is then reduced to

$$\int_0^{t_f}\{x_1(t)[\ \alpha(t) - y_{11}(t)] + \dot{x}_1(t)[\beta(t) -\ \dot{y}_{11}(t)]\}dt = 0 \quad (64)$$

so by Lemma 1, $\beta(t) -\ \dot{y}_{11}(t)$ is differentiable on $[0, t_f]$ with:

$$\frac{d}{dt}[\beta(t) -\ \dot{y}_{11}(t)] = \alpha(t) - y_{11}(t) \quad (65)$$

Simplifying (65) further gives:

$$\ddot{y}_{11}(t) - y_{11}(t) = \dot{\beta}(t) -\ \alpha(t) \quad (66)$$

Substituting (62) and (63) in (66) leads to:

$$\ddot{y}_{11}(t) - y_{11}(t) =\ \mu\ddot{x}_2(t) - [\left(\tfrac{1}{2}Q + \mu C_1^2\right)x_2(t) + (H - 2\mu C_1)\dot{x}_2(t) + 2\mu C_1C_2w_2(t) - 2\mu C_2\dot{w}_2(t) + \mu C_2^2x_2(t)]. \quad (67)$$

On solving for $y_{11}(t)$ in (67) and eliminating the resulting constants gets a tidier form:

$$y_{11}(t) =\ -\mu\dot{x}(0)Sinh(t_f) + \int_0^{t_f}\mu\dot{x}(s)Sinh(t_f - s)ds - \int_0^{t_f}Sinh(t_f - s)\,[\left(\tfrac{1}{2}Q + \mu C_1^2 + C_2^2\right)x(s) + (H - 2\mu C_1)\dot{x}(s) + 2\mu C_1C_2x(s+r) - 2\mu C_2\dot{x}(s+r)]ds$$
$$+ [\left(\tfrac{1}{2}Q + \mu C_1^2 + C_2^2\right)x(0) + (H - 2\mu C_1)\dot{x}(0) + 2\mu C_1C_2x(r) - 2\mu C_2\dot{x}(s+r)]Cosh(t) + \frac{Sinht}{Sinht_f}\{\left(\tfrac{1}{2}Q + \mu C_1^2 + C_2^2\right)x(t_f) + (H - 2\mu C_1)\dot{x}(t_f) + 2\mu C_1C_2x(t_f - r) - 2\mu C_2\dot{x}(t_f - r) -\ [\left(\tfrac{1}{2}Q + \mu C_1^2 + C_2^2\right)x(0) + (H - 2\mu C_1)\dot{x}(0) + 2\mu C_1C_2x(r) - 2\mu C_2\dot{x}(r)]Cosh(t_f)\ + \int_0^{t_f}Sinh(t_f - s)[\left(\tfrac{1}{2}Q + \mu C_1^2 + C_2^2\right)x(s) + (H - 2\mu C_1)\dot{x}(s) + 2\mu C_1C_2x(s+r) - 2\mu C_2\dot{x}(s+r)]ds + \mu\dot{x}(0)Sinh(t_f) - \int_0^{t_f}\mu\dot{x}(s)Cosh(t_f - s)ds\}\ ;\ 0 \le t \le t_f \quad (68)$$

Now from (61), we readily obtain:

$$y_{21}(t) = -2\mu D\dot{x}(t);\ \ 0 \le t \le t_f \quad (69)$$

$$y_{31}(t) =\ 2\mu C_1C_2x(t+r) - 2\mu C_2\dot{x}(t+r) + \mu C_2^2x(t)\ ;\ -r \le t \le 0. \quad (70)$$

Thus, the first column of the operator, G_{11}, G_{21}, and G_{31} are uniquely determined considering (50).The second column of the operator, G_{12}, G_{22}, and G_{32}, is obtained by repeating the same line of argument setting $x_2(t) \equiv h_2(t) = 0$, then $\dot{x}_2(t) = 0 = x_2(t-r)$, and the last column of the operator $G_{13},\ G_{23}$, and G_{33} is arrived at by setting $x_2(t) \equiv u_2(t) = 0$, for $0 \le t \le t_f$. This implies that, $x_2(t) = 0$ and $x_2(t-r) = h_2(t)$. With this, the proof of Theorem 1 is concluded hence, the application of the constructed control operator in the CGM algorithm in CLRP will be considered in the next section. ■

IV. COMPUTATIONAL IMPLEMENTATION

Based on the construction of the operator, this paper is focused on the introduction of the constructed linear operator to ECGM algorithm with a view to solving the following Bolza form CLRP with delay and without the delay in the state equation. Sequel to the sixth step of the ECGM, the terminating criteria can be set as the function is said to converge when the gradient value tends to zero. In another vein, the value of the Gradient Norm can be used as criterion in determining the convergence of the function as the gradient norm tends to zero. Also, analytical results or existing results can be used as bases of comparison with the results from function value applying the operator in the ECGM.

However, for the purpose of this work, two or more of the terminating criteria will be used to determining the convergence of the test problems and the penalty parameter varied from 100, 10, 1.0, 0.1 and 0.01 in each of the problems. The following problems were tested:

Problem 1:

$$Min\,J = \tfrac{1}{2}x(1)^THx(1) + \tfrac{1}{2}\int_0^1\{x^T(t)A(t)x(t) + u^T(t)Bu(t)\}dt$$

Subject to the delay differential equation:
$\dot{x}(t) = 2x(t) + 3x(t-2) + 5u(t)$,
where $A(t) = 1$, $H(t) = 1$, $B = 1$, $\mu = 10$, $x_0 = 5$ and $u_0 = 3$.

Problem 2:
The first order differential delay system
$\dot{x}(t) = -10x(t) + x(t-0.3) + u(t)$
is to be controlled to minimize the cost function:
$J = \frac{1}{2}x^2(0.04) + \int_0^{0.04}\left[\frac{1}{4}x^2(t) + \frac{1}{2}u^2(t)\right]dt.$
Neither the admissible state nor the control values is constrained by any boundaries.

Table 1: Table of result of Problem 1 at $\mu = 1.0$

It	$x(t)$	$u(t)$	Function Value	Gradient Norm
0	5	3	1185.5	480.985447
1	3.807782264	2.471980315	342.1383314	58.4011412
2	3.783607526	1.947076891	93.683661529	7.26522083
3	3.281234698	1.436695136	15.676671954	0.90761723
4	2.170522883	1.146689162	6.1676563821	1.13372e-03
5	1.783292874	0.731936807	0.3676562134	1.42935e-07
6	1.276892874	0.102174606	2.6562195e-01	1.77106e-08
7	0.783228745	1.7468926e-02	3.1676576e-03	3.47162e-10

Table 2: Table of result of Problem 1 at $\mu = 10.0$

It	$x(t)$	$u(t)$	Function Value	Gradient Norm
0	5	3	15269	4488.42222
1	3.755579882	2.123826653	832.7194001	3.43971e-04
2	2.858122895	1.513066012	218.9718767	1.20733e-04
3	2.090247123	0.951291644	19.64323098	7.33610e-05
4	1.170205352	0303501193	12.83088067	6.34701e-05
5	-0.76898843	-1.06183879	2.310188036	1.07284e-06
6	-1.45870947	-1.54745871	0.665029513	3.17275e-07
7	-1.84640129	-1.82042594	0.101323441	1.53914e-08

Table 3: Table of Results for the Function Values at Variant Penalty Constant Values

It	Function Value at μ = 100	Function Value at μ = 10.0	Function Value at μ = 1.0	Function Value at μ = 0.1	Function Value at μ = 0.01
0	824.1859211	583.0692049	421.9027063	258.1976622	167.9510533
1	723.5690201	501.6732892	398.1669036	233.9603407	118.4820256
2	439.2075418	383.5905114	174.3962763	189.3810394	26.92076553
3	267.3819374	138.2973453	89.75542937	62.20669252	7.273702612
4	138.1803677	53.92887113	22.86491427	13.83552049	3.161729472
5	69.57107254	7.938255916	9.368827461	5.265427532	1.826405386
6	26.19851035	0.962448105	4.285937585	0.639025825	0.389052832
7	15.94275483	0.269584227	2.968352184	0.295527681	5.837286551
8	6.286449173	0.082775391	0.965824104	0.075249165	7.962118362
9	2.839461197	-0.392809553	0.583663163	0.009286319	29.37558326
10	0.933827459	-5.832767224	0.376592734	0.003817684	83.92753358
11	0.284371155	-17.27945316	-2.683533965	0.001943852	31.78595316
12	7.293639912	-28.25691174	-9.935082417	8.285128e-05	9.538166489
13	7.293639912	-56.38817253	-37.18360934	8.285128e-05	-26.95373721
14	7.293639912	-6.922573941	-8.692184336	8.285128e-05	-9.381426935
15	7.293639912	-2.530926412	-2.928558324	8.285128e-05	-3.916553745
16	7.293639912	-0.392809553	0.966280834	8.285128e-05	0.005295722
17	7.293639912	-0.392809553	0.966280834	8.285128e-05	0.000091536
18	7.293639912	-0.392809553	0.966280834	8.285128e-05	3.68475e-07
19	7.293639912	-0.392809553	0.966280834	8.285128e-05	3.68475e-07
20	7.293639912	-0.392809553	0.966280834	8.285128e-05	3.68475e-07

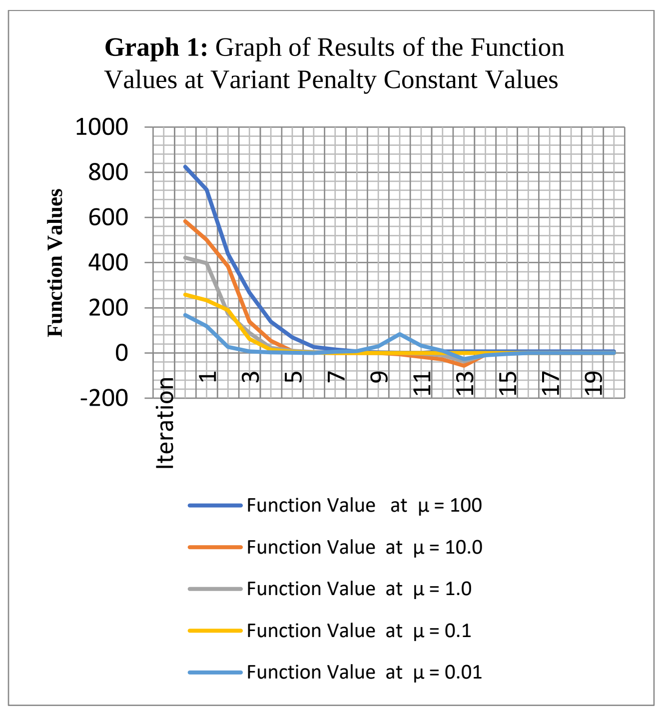

Figure 1: Graph for Problem 1

The method was tested on a number of problems with varying penalty constants. The Figure 1 depicts the penalty constant increases the early iteration function values are seen to

It	Gradient Norm at μ = 100	Gradient Norm at μ = 10.0	Gradient Norm at μ = 1.0	Gradient Norm at μ = 0.1	Gradient Norm at μ = 0.01
0	21869.7938	3187.83328	398.246155	29.4550505	11.6736455
1	2620.25064	1169.11338	55.7063362	0.37171826	2.15188036
2	441.906755	519.236774	16.6711739	0.030718178	0.35574116
3	83.9642867	254.696288	1.66303054	0.001678497	0.044643318
4	16.3358616	131.616436	0.42433319	0.000307744	0.015536584
5	3.19212051	69.9392691	0.05363454	7.94041e-06	0.000929407
6	0.62427367	37.7312857	0.01006791	2.79453e-06	0.000462507
7	0.12210688	20.5241102	0.001829325	4.32958e-08	2.78093e-05
8	0.023884643	11.2146842	0.000239653	1.84264e-08	8.89814e-06
9	0.004671975	6.14305157	6.00744e-05	3.19832e-10	1.14613e-06
10	0.000913865	3.36954157	6.13528e-06	8.85661e-11	1.60913e-07
11	0.000178757	1.84961429	1.74487e-06	3.01731e-12	4.90613e-08
12	3.49668e-05	1.01570899	1.78222e-07	3.99493e-13	3.23605e-09
13	6.83955e-06	0.55789835	4.41292e-08	3.07928e-14	1.60231e-09
14	1.33786e-06	0.30647461	5.82197e-09	1.81894e-15	8.95999e-11
15	2.61693e-07	0.16836949	1.04304e-09	3.13128e-16	3.25268e-11
16	5.11886e-08	0.09250144	1.98685e-10	8.55939e-18	3.55793e-12
17	1.00128e-08	0.05082091	2.49666e-11	2.88933e-18	5.84319e-13
18	1.95856e-09	0.02792168	6.44873e-12	4.58139e-20	1.54242e-13
19	3.83106e-10	0.01534062	6.48195e-13	1.96884e-20	1.13915e-14
20	7.49378e-11	0.00842841	1.83747e-13	3.29435e-22	5.44637e-15

decrease considerably while with passing of time irrespective of the penalty constant used all the function values tend toward the same point.

V. CONCLUSION

It follows from the tables that, while authors in [1] and [2] constructed control operators for Continuous-Time Linear Regulator, the authors in [2] focused on the same form of control problem with time lag in the state equation. This paper constructed a control operator, G, for the Bolza type of the CLRP which has helped to bridge the gap in Mayer's, Lagrange's, and Bolza's form of the CLRP with lag or without delay parameter in the constraint. This has led to an increased range of problems that the ECGM algorithm could be used to solve.

With the numerical experiment of the linear operator show cased here, it makes the development of the linear operator useful, relevant, robust and efficient as, it is meant to fix all forms of CLRP either with delay or without time lag in the state variable with little or no effect on the penalty parameter as it is being varied.

Table 4: Gradient Norm of Problem 2 at Variant Penalty Constant Values

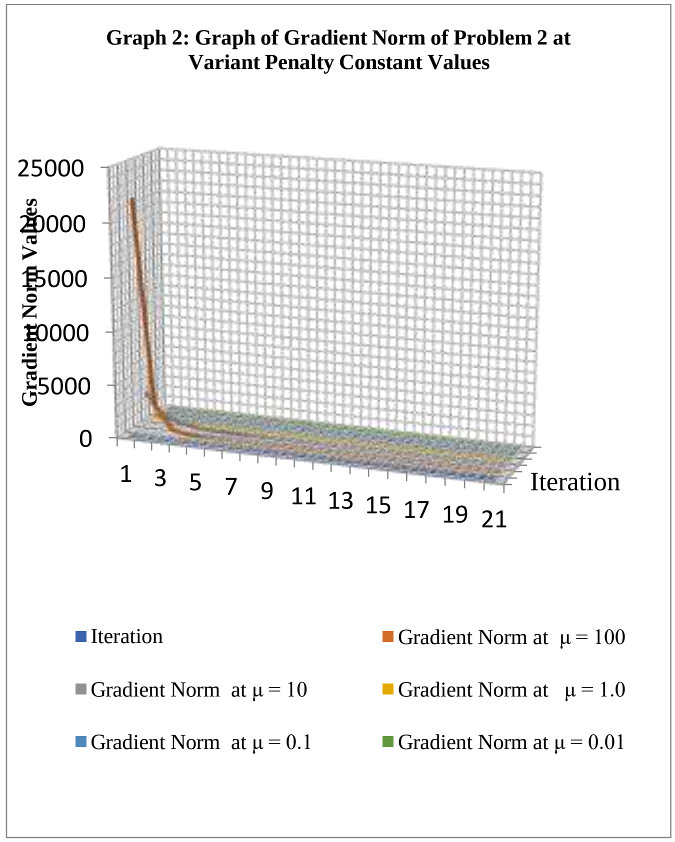

Figure 2: Graph for Problem 2.

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