Approximate computation of third-order dispersive partial differential equation with Caputo fractional derivative

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Abstract: *The goal of this paper is to approximate fractional third-order dispersive partial differential equations using an efficient scheme titled as Reduced differential transform method (RDTM). The advantage of using RDTM is, it can produce an analytically approximate answer in the form of a convergent power series with easily ascertainable components. Without using any discretization, constrictive assumptions or transformation, the approach determines the solution while taking into account the application of the proper beginning conditions. Our test cases show the precision and effectiveness of the suggested approach, and the solution behavior is shown in tables. The numerical findings on different values of α are contrasted with the Differential Transform Method, Laplace-Adomian Decomposition and Homotopy Analysis Sumudu Transform method. Additionally, it has been found that there is a strong correlation between the numerical results and the documented numerical and precise solutions in this study. In order to conveniently explain many additional fractional differential equations, the offered approach thus exhibits the reliability, efficacy, competency, and strengthening of resultant conclusions.*

Keywords: *Reduced differential transform method; fractional third-order dispersive partial differential equation (FTD-PDE); fractional differential equations*

1 Introduction

There are great advancements expected in the field of fractional order calculus and its applications. Numerous academicians have utilized FDEs to explore and depict logical occurrences across a range of scientific disciplines. [1-5]. Viscoelasticity, electromagnetic waves, the diffusion equation, and other areas are some of the most important applications [6]. The nonlocality of FDEs is their most important characteristic when applied to the aforementioned phenomena and others. Thus, differential operators offer a great tool for describing memory and hereditary features of diverse materials and procedures. In the context of fractional derivatives, partial differential equations (PDEs) are regarded as a potent tool in mathematical modelling to comprehend and interpret some structures of physical events that are intricate and unpredictable owing to outside influences. For this reason, scholars have used them to both construct a natural problem that is easily accessible and to simplify the regulating design without sacrificing any genetic information or memory impact. Additionally, several efforts have been effective in recommending solid numerical methods for treating the fractional PDEs of physical importance. Numerous real-world issues, such as traffic flow, oscillation, earthquakes, and gas dynamics [1] which may be restated as nonlinear PDEs in light of fractional derivatives, can be understood very well through the solutions of PDEs of fractional order. Therefore, it is important to develop a practical and useful method for identifying analytical answers to these and other challenges. A variety of iterative strategies, for instance variational iteration method, homotopy perturbation method, modified form of homotopy perturbation transform technique, homotopy analysis scheme, finite difference scheme, residual power series approach, the Adomian decomposition method, Differential transform technique , predictor-corrector, Haar wavelet and numerous others, have been constructed to find the solutions of a variety of Fractional models including both ODEs and PDEs [7-18]

The method named "Reduced differential transform method" (RDTM) was created by Keskin and Oturanc [19] and it was shown that it is the analytical methodology that can be employed as the simplest and it delivers the precise answer to differential models. RDTM strategy is very trustworthy, efficient, and potent computational approach for resolving physical issues [20-22]. In chemical reactions the Brusselator reaction-diffusion system arises which was studied by Taghavi.et.al as a time fractional dynamical model using RDTM [23]. Ramani and his fellows determines approximate solutions of Fitzhugh-Nagumo equation through RDTM [24]. The key role of this work is to analyze FTD-PDEs by the proposed RDTM technique.

2 Preliminaries of Fractional Calculus

Numerous definitions on fractional derivatives and integrals are available in literature. Some of them are $\mathcal{R}$eimann–$\mathcal{L}$iouville, $\mathcal{J}$umarie's , $\mathcal{C}$aputo and $\mathcal{R}$iesz, Grunwald- Letnikov, the Weyl and many others. The widely used are Caputo and $\mathcal{R}$eimann- $\mathcal{L}$iouville fractional derivative.

Definition 1 [25] The Reimaan –Liouville defined the fractional integral operator for an integrable function $\wp(x) \in S_\mu$ as

$$\begin{cases} J_t^{\alpha}\wp(x) = \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^x (x-t)^{\alpha-1}\wp(t)dt, \qquad where \quad \alpha > 0, x > 0 \\ J_t^0\wp(x) = \wp(x). \end{cases}$$

Some properties are listed below:

i. $J^{\alpha}J^{\gamma} = J^{\alpha+\gamma}$

ii. $J^{\alpha}J^{\gamma} = J^{\gamma}J^{\alpha}$

iii. $\left(J^{\alpha}J^{\gamma}\right)\wp(t) = \left(J^{\gamma}J^{\alpha}\wp\right)(t)$

iv. $J^{\alpha}\left(t-\alpha\right)^{\eta} = \dfrac{\Gamma(\eta+1)}{\Gamma(\alpha+\eta+1)}\left(t-\alpha\right)^{\alpha+\eta}$

where $\alpha, \gamma \geq 0$ *and* $\eta > -1$.

Definition 2 [25]

Caputo defined the fractional order derivative as

$$D_t^{\alpha}\wp(x) = J_t^{r-\alpha}D_t^r\wp(x) = \frac{1}{\Gamma(r-\alpha)}\int_0^x (x-t)^{r-\alpha-1}\wp^{(r)}(t)\,dt,$$

$$where \quad r-1 < \alpha \leq r, r \in \mathrm{N}, \; x > 0.$$

For more details on fractional derivatives, refer to [26].

3 Reduced Differential Transform Method

A constantly differentiable function $s(x,t) = p(x).q(t)$ can be signified as [22]

$$s(x,t) = \left(\sum_{i=0}^{\infty} P(i)x^i\right)\left(\sum_{j=0}^{\infty} Q(j)t^j\right) = \left(\sum_{k=0}^{\infty} S_k(x)t^k\right) \tag{1}$$

The function $s(x,t)$ is transformed into

$$S_k(x) = \frac{1}{\Gamma(\alpha k+1)}\left(\frac{\partial^{\alpha k}}{\partial t^{\alpha k}}s(x,t)\right)_{t=t_0} \tag{2}$$

However, when applying inverse transformation on $S_k(x)$, it yields

$$s(x,t)=\sum_{k=0}^{\infty}S_k(x)(t-t_0)^{\alpha k} \tag{3}$$

From equations (2) and (3), and $t_0=0,$ we get

$$s(x,t)=\sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha k+1)}\left(\frac{\partial^{\alpha k}}{\partial t^{\alpha k}}s(x,t)\right)_{t=0}t^{\alpha k}. \tag{4}$$

For more details, see Table 1

Table 1: Specific operations of RDTM

Functional Form	Transformed Form
$s(x,t)$	$S_k(x)=\frac{1}{\Gamma(\alpha k+1)}\left(\frac{\partial^{\alpha k}}{\partial t^{\alpha k}}s(x,t)\right)_{t=t_0}$
$s(x,t)=u(x,t)\pm v(x,t)$	$S_k(x)=U_k(x)\pm V_k(x)$
$s(x,t)=\frac{\partial^{N\alpha}}{\partial t^{N\alpha}}u(x,t)$	$S_k(x)=\frac{\Gamma(\alpha k+N\alpha+1)}{\Gamma(\alpha k+1)}U_{k+N}(x)$
$s(x,t)=x^m t^n$	$S_k(x)=x^m\sigma(k\alpha-n)$
$s(x,t)=x^m t^n u(x,t)$	$S_k(x)=x^m\sigma U(k\alpha-n)$
$s(x,t)=u(x,t).v(x,t)$	$S_k(x)=\sum_{r=0}^{k}V_r(x)U_{k-r}(x)=\sum_{r=0}^{k}U_r(x)V_{k-r}(x)$
$s(x,t)=\frac{\partial^m}{\partial x^m}u(x,t)$	$S_k(x)=\frac{\partial^m}{\partial x^m}U_k(x)$
$s(x,t)=cu(x,t)$	$S_k(x)=cU_k(x)$

3.1 Method Implementation

Considering the general form of fractional order partial differential equation.

$$L(s(x,t))+R(s(x,t)+\mathrm{N}(s(x,t))=L(u(x,t)) \tag{5}$$

with the initial guesses

$$s(x,0)=v(x),\ s_t(x,0)=w(x) \tag{6}$$

Here $L = D_t^{\alpha}, R$ is the linear differential operator, $u(x,t)$ denotes non-homogeneous source term, whereas N is the generalized nonlinear operator.

By applying the transformed forms of RDTM [22] in equations (5) and (6), we get

$$\frac{\Gamma(\alpha k+\alpha+1)}{\Gamma(\alpha k+1)} S_{k+1}(x) = U_k(x) - R(S_k(x)) - N(S_k(x)) \quad (7)$$

$$S_0(x) = v(x),\ S_1(x) = w(x) \quad (8)$$

After successive iterations of $S_{k+1}(x)$, we apply the inverse transformation to get the series solution $s(x,t) = \sum_{k=0}^{n} S_k(x) t^{\alpha k}$. Additionally, the exact solution is given by $s(x,t) = \lim_{n\to\infty} s_n(x,t)$.

4 Application

Here we consider two model problems of FTD-PDE to elucidate the competence and reliability of the proposed method.

Test Problem 1 Consider a FTD-PDE [16]

$$\omega_t^{\alpha}(x,t) + 2\omega_x(x,t) + \omega_{xxx}(x,t) = 0, \qquad t>0, \quad 0<\alpha\le 1 \quad (9)$$

with an initial guess $\omega(x,0) = \sin x$ and exact solution at $\alpha = 1$ is $\omega(x,t) = \sin(x-t)$.

Test Problem 2 Consider another FTD-PDE [16]

$$\omega_t^{\alpha}(x,t) + \omega_{xxx}(x,t) = -\sin\pi x \sin t - \pi^3 \cos\pi x \cos t, \qquad t>0, \quad 0<\alpha\le 1 \quad (10)$$

with an initial guess $\omega(x,0) = \sin\pi x$ and time- dependent boundary conditions

$\omega(0,t) = 0,\ \omega_x(0,t) = \pi\cos t,\quad \omega_{xx}(0,t) = 0$

The exact solution at $\alpha = 1$ is $\omega(x,t) = \sin\pi x \cos t$.

5 Results and Discussion

Table 2 and 3 shows the results obtained of the test problems 1 and 2 respectively, indicating the approximate result obtained by RDTM at various grid points. The proposed method RDTM is compared with other techniques namely LADM, HASTM and DTM revealing high accuracy and precision. Moreover, the approximate solution derived by RDTM and the actual solution for $\alpha = 1$ are in good agreement. Graphical illustrations are also given in Fig. 1-2 and 3-4 for problem 1 and 2 respectively at $\alpha = 1$.

Table 2. Contrast with LADM, HASTM, DTM, RDTM and Exact solution of Problem 1 for $\alpha = 1$

x	t	LADM [7]	HASTM [9]	DTM [16]	RDTM	Exact
0	0.1	−0.0998334	−0.099833	−0.0998334	−0.0998334	−0.0998334
0.2	0.1	0.0998334	0.0998334	0.0998334	0.0998334	0.0998334
0.4	0.1	0.2955202	0.2955202	0.2955202	0.2955202	0.2955202
0.6	0.1	0.4794255	0.4794255	0.4794255	0.4794255	0.4794255
0.8	0.1	0.6442176	0.6442176	0.6442176	0.6442176	0.6442176
1.0	0.1	0.7833269	0.7833269	0.7833269	0.7833269	0.7833269

Table 3. Contrast with LADM, HASTM, DTM, RDTM and Exact solution of Problem 2 for $\alpha = 1$

x	t	LADM [7]	HASTM [9]	DTM [16]	RDTM	Exact
0	0.1	0	0	0	0	0
0.2	0.1	0.5877852	0.5877852	0.5877852	0.5877852	0.5848487
0.4	0.1	0.9510565	0.9510565	0.9510565	0.9510565	0.9463051
0.6	0.1	0.9510565	0.9510565	0.9510565	0.9510565	0.9463051
0.8	0.1	0.5877852	0.5877852	0.5877852	0.5877852	0.5848487
1.0	0.1	0	0	0	0	0

Table 4. Numerical values of the problem 1 by RDTM for different values of α

x	t	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 0.99$	$\alpha = 1$
0	0.1	-0.333988034	-0.191293221	-0.102574207	−0.099833416
0.2	0.1	-0.147534765	0.006496338	0.097081289	0.099833416
0.4	0.1	0.0448002487	0.204026911	0.292866461	0.295520207
0.6	0.1	0.2353492187	0.393423574	0.476975971	0.479425539
0.8	0.1	0.4165155578	0.567135681	0.642069955	0.644217688
1.0	0.1	0.5810767360	0.718237878	0.781566635	0.783326911

Moreover, numerical values on different values of α are also examined and listed in table 4 and 5 respectively. Both test problems were analyzed at different grid points and It can be clearly seen that as the value of α comes close to 1, the numerical results are approaching towards the exact solution. It is notable that RDTM requires less computation and gives an approximate series solution in just few iterations.

Table 5. Numerical values of the problem 2 by RDTM for different values of α

x	t	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 0.99$	$\alpha = 1$
0	0.1	0	0	0	0
0.2	0.1	0.5877852	0.5877852	0.5877852	0.5877852
0.4	0.1	0.9510565	0.9510565	0.9510565	0.9510565
0.6	0.1	0.9510565	0.9510565	0.9510565	0.9510565
0.8	0.1	0.5877852	0.5877852	0.5877852	0.5877852
1.0	0.1	0	0	0	0

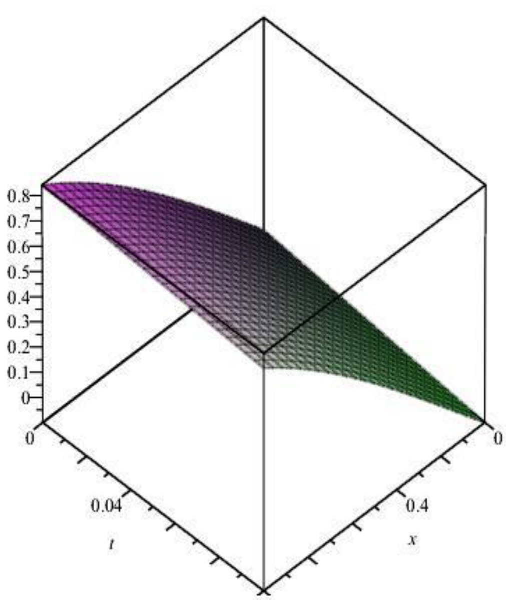

Fig 1 Exact solution for problem 1 at $\alpha = 1$

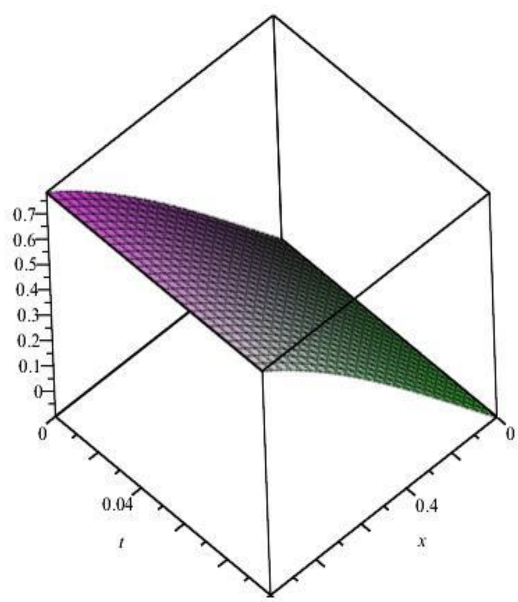

Fig 2 RDTM solution for problem 1 at $\alpha = 1$

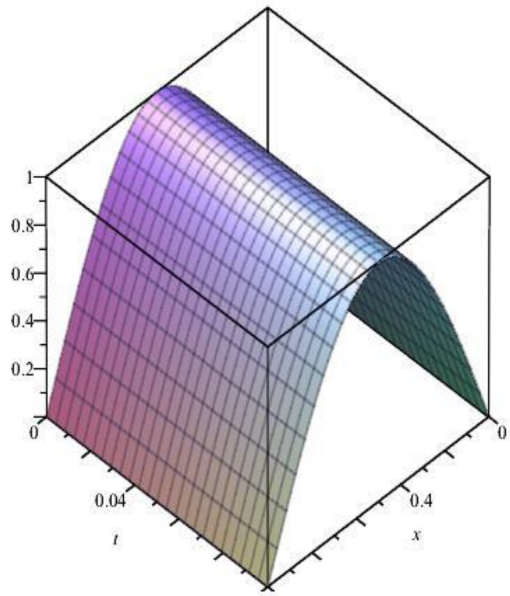

Fig 3 Exact solution for problem 2 at $\alpha = 1$

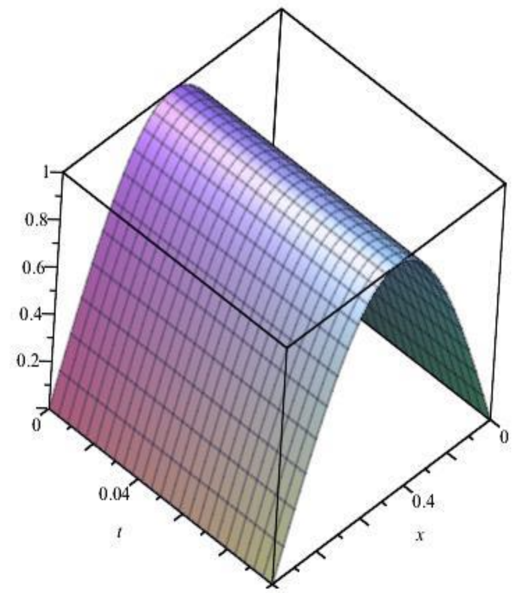

Fig 4 RDTM solution for problem 2 at $\alpha = 1$

6 Conclusion

In this paper, time-fractional third-order dispersive partial differential equations are analyzed by employing RDTM technique in order to achieve the series as well as approximate analytic solutions. Two test problems are analyzed at distinct values of α to access the competence of the proposed method. The findings obtained by the other approaches and the precise results by RDTM exhibit a very high degree of agreement. The numerical outcomes demonstrate high convergence rate of the suggested method. RDTM may be summed up as a great improvement over current numerical methods and may therefore be utilized to resolve several non-linear fractional partial differential equations that arises in scientific and engineering applications.

7 Conflicts

The authors certify that they do not have any conflicting interests to declare.

References

1. Kilbas, A.A., H.M. Srivastava, and J.J. Trujillo, *Theory and applications of fractional differential equations*. Vol. 204. 2006: elsevier.
2. Ross, B. *A brief history and exposition of the fundamental theory of fractional calculus*. in *Fractional Calculus and Its Applications: Proceedings of the International Conference Held at the University of New Haven, June 1974*. 2006. Springer.
3. Hilfer, R., *Applications of fractional calculus in physics*. 2000: World scientific.
4. Mohan Jena, R., S. Chakraverty, and M. Yavuz, *Two-hybrid techniques coupled with an integral transformation for Caputo time-fractional Navier-stokes equations.* Progress in Fractional Differentiation & Applications, 2020. 6(3): p. 201-213.
5. Wang, K.-L., K.-J. Wang, and C.-H. He, *Physical insight of local fractional calculus and its application to fractional Kdv–Burgers–Kuramoto equation.* Fractals, 2019. 27(07): p. 1950122.
6. Miller, K.S. and B. Ross, *An introduction to the fractional calculus and fractional differential equations.* (No Title), 1993.
7. Shah, R., et al., *Application of Laplace–Adomian decomposition method for the analytical solution of third-order dispersive fractional partial differential equations.* Entropy, 2019. 21(4): p. 335.
8. Nadeem, M. and J.-H. He, *He–Laplace variational iteration method for solving the nonlinear equations arising in chemical kinetics and population dynamics.* Journal of mathematical chemistry, 2021. 59: p. 1234-1245.
9. Pandey, R.K. and H.K. Mishra, *Homotopy analysis Sumudu transform method for time—fractional third order dispersive partial differential equation.* Advances in Computational Mathematics, 2017. 43: p. 365-383.
10. Zeidan, D., et al., *Mathematical studies of the solution of Burgers' equations by Adomian decomposition method.* Mathematical Methods in the Applied Sciences, 2020. 43(5): p. 2171-2188.
11. Agbata, B., et al., *Analysis of homotopy perturbation method (HPM) and its application for solving infectious disease models.* International Journal of Mathematics and Statistics Studies, 2021. 9(4): p. 27-38.
12. Iqbal, N., et al., *Analytical analysis of Fractional-Order Newell-Whitehead-Segel equation: A modified homotopy perturbation transform method.* Journal of Function Spaces, 2022. 2022: p. 1-10.
13. Naik, P.A., J. Zu, and M. Ghoreishi, *Estimating the approximate analytical solution of HIV viral dynamic model by using homotopy analysis method.* Chaos, Solitons & Fractals, 2020. 131: p. 109500.
14. Cui, M., *Finite difference schemes for the two-dimensional multi-term time-fractional diffusion equations with variable coefficients.* Computational and Applied Mathematics, 2021. 40(5): p. 167.
15. Hasan, S., et al., *Solution of fractional SIR epidemic model using residual power series method.* Applied Mathematics and Information Sciences, 2019. 13(2): p. 153-161.
16. Kanth, A.R. and K. Aruna, *Solution of fractional third-order dispersive partial differential equations.* Egyptian journal of basic and applied sciences, 2015. 2(3): p. 190-199.

17. Zabidi, N.A., et al., *Numerical solution of fractional differential equations with Caputo derivative by using numerical fractional predict–correct technique.* Advances in Continuous and Discrete Models, 2022. 2022(1): p. 1-23.
18. Shah, K., et al., *Haar wavelet collocation approach for the solution of fractional order COVID-19 model using Caputo derivative.* Alexandria Engineering Journal, 2020. 59(5): p. 3221-3231.
19. Keskin, Y. and G. Oturanc, *Reduced differential transform method for partial differential equations.* International Journal of Nonlinear Sciences and Numerical Simulation, 2009. 10(6): p. 741-750.
20. Abazari, R. and A. Kılıçman. *Numerical study of two-dimensional Volterra integral equations by RDTM and comparison with DTM*. in *Abstract and Applied Analysis*. 2013. Hindawi.
21. Gubes, M. and G. Oturanc, *Approximate solutions of coupled Ramani equation by using RDTM with compared DTM and exact solutions.* New Trends in Mathematical Sciences, 2016. 4(4): p. 198-212.
22. Gusu, D.M., et al., *Fractional order Airy's type differential equations of its models using RDTM.* Mathematical Problems in Engineering, 2021. 2021: p. 1-21.
23. Taghavi, A., A. Babaei, and A. Mohammadpour, *Analytical approximation solution of a mathematical modeling of reaction-diffusion Brusselator system by reduced differential transform method.* Journal of Hyperstructures, 2015. 3(2).
24. Ramani, P., et al., *Approximate analytical solution for non-linear Fitzhugh–Nagumo equation of time fractional order through fractional reduced differential transform method.* International Journal of Applied and Computational Mathematics, 2022. 8(2): p. 61.
25. Altaf, S. and S.Y. Khan, *Numerical Solution of Fractional Electrical Circuits by Haar Wavelet.* Matematika, 2019. 35.
26. Podlubny, I., *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*. 1998: Elsevier.