Study on locally compact Abelian groups equidimensional with their duals

Jawaid Ahmed Khan*, Mushtaque Hussain**

*Department of Mathematics, Sir Syed University of Engineering & Technology, Karachi

**Department of Mathematics, Government College University, Hyderabad

ABSTRACT

Using the generalized dimension theory of topological spaces for locally compact Abelian topological groups, the conditions were investigated so that the dimension of an arbitrary locally compact abelian group may equal the dimension of its dual locally compact Abelian group. While these dimension functions show all kinds of topological behavior in arbitrary topological spaces, they behave well for separable metric spaces and for locally compact topological groups where these dimension functions coincide and can be used interchangeably.

Key words: Abelian group, Equidimensional, Locally compact.

1. INTRODUCTION

The intuitive notion of dimension in Euclidean spaces R^n $(n = 1,2,3)$ has led to three main theories of dimension of general topological spaces. A topological space X has strong inductive dimension less than or equal to , written $Ind\, X \leq \mathrm{n}$, if for any disjoint closed sets F and G of X, there exists an open set U such that $F \subset U \subset \; X - G$ and Ind boundary $U \leq n - 1$. If $Ind\, X \; \leq n$ is true and $Ind\, X \leq n - 1$ is false, then $Ind\, X \; = n$. If $Ind\, X \; \leq n$ is false for each n, then $Ind\, X \; = \infty$. If X is empty, $Ind\, X \; = -1$. X has weak inductive dimension less than or equal to n, written $Ind\, X \leq n$, if for every neighborhood $U(x)$ of every point $x \in X$, there exists an open neighborhood $V(x)$ such that $x \in V(x) \subset U(x)$ and ind boundary $V(x) \leq n - 1$. If $ind\, X \; \leq \; n$ is true and $ind\, X \; \leq n - 1$ is false, then $ind\, X \; = \; n$. If $ind\, X \; \leq \; n$ is false for each n , then $ind \; X \; = \; \infty$. If X is empty, $ind\, X \; = \; -1$. If for any finite open covering O_1 , of topological space X these exists an open refinement O_2 such that order $O_2 \; \leq n + 1$, then X has covering dimension less than or equal to n , written $dim\, X \leq n$. X has covering dimension n if $dim\, X \leq n$ is true but $dim\, X \leq n - 1$ is false. If $dim\, X \leq n$ is false for each n , then $dim\, X = \infty$. If X is empty, $dim\, X = -1$. [1,2].

While these dimension functions show all kinds of topological behavior in arbitrary topological spaces, they behave well for separable metric spaces and for locally compact topological groups where these dimension functions coincide and can be used interchangeably [3]. As such we shall denote the dimension of a locally compact abelian (LCA) group G as dimension G or briefly as G .

Every LCA group G has a dual LCA group $\hat{G}$, the group of all continuous homomorphism $G \to T$. For each closed subgroup H of G , $A(\hat{G}, H)$ is the closed subgroup of annihilators of H in $\hat{G}$. $t(G)$, $b(G)$ and G_0 denote the maximal torsion subgroup, the subgroup of compact elements of G and the connected component of the zero element in G. $\cong$ denotes topological isomorphism. $H \leq G$ means H is a subgroup of G. We shall often use : If H is a closed subgroup of an LCA group G, then $dim\, G = dim\, H + dim\,(G/H)$. [4-6].

For a given LCA group G, $dim\, G$ may or may not be equal to $dim\, \hat{G}$. For example, $G = R^n$ $(n \geq 1)$, then $\hat{G} = R^n$, and so $dim\, G = dim\, \hat{G} = n$. On the other hand, let $G = R^n \times T$ then $\hat{G} = R^n \times Z$ where Z is the disconnect group of all integers. Now $dim\, G = n + 1$ but $dim\, \hat{G} = n$, so $dim\, G$ is not equal to $dim\, \hat{G}$ in this case.

DEFINITION 1.1

We shall say that an LCA group is a $dn - group$ if $dim\, G = dim\, \hat{G}$.

Thus R^n is a $dn - group$ but $R^n \times T$ is not . Since dual of $\hat{G}$ is G , G is a $dn - group$ iff $\hat{G}$ is a $dn - group$.

2. LCA DN-GROUPS

We shall now investigate as to which LCA groups are $dn - groups$ and which are not. We recall that an LCA group G is called self dual if $G \cong \hat{G}$. For each LCA group G , $G \times \hat{G}$ is self dual, for ($G \times \hat{G}$)^ $\cong \hat{G} \times G \cong G \times \hat{G}$. Also if G_1 , G_2 are self-dual, then so is

$G_1 \times G_2$, for $(G_1 \times G_2)^\wedge = \hat{G}_1 \times \hat{G}_2 = G_1 \times G_2$.

Our next theorem provides wealth of $dn - groups$.

THEOREM 2.1

Let G be an LCA group. If G is self-dual, then G is a $dn - group$. The converse is not true.

Proof.

Since G is self-dual, there exists a topological isomorphism $f: G \rightarrow \hat{G}$ (i.e f is one-to-one, continuous, open homomorphism of G onto $\hat{G}$). Since $dim\, G$ is a topological invariant ([6], 33.1), this implies $dim\, G = dim\, \hat{G}$. For the converse, see example 2.3.

In the next Corollary, we present several families of self-dual LCA groups. All these are, of course, $dn - groups$ by our theorem 2.1.

COROLLARY 2.2

The following are self-dual groups:

(i) $R^n (n \geq 1)$.

(ii) All finite discrete abelian groups.

(iii) For each $i \in I$, where I is an index set, let G_i be a self-dual LCA group with topological isomorphism f_i of G_i onto $\hat{G}_i$. Suppose also that G_i has a compact open subgroups H_i such that f_i (H_i) is the annihilator of H_i in $\hat{G}_i$. The local direct product G of the G_i relative to the compact open subgroups H_i is self-dual.

(iv) Any finite product $F_{p_1} \times F_{p_2} \times \cdots \cdots \times F_{p_n}$, where for each prime p, F_p is the LCA group of $p - adic$ numbers.

(v) Any LCA group of the form $R^n \times G \times \hat{G}$.

Proof.

For proofs of (i-iii) ([5], 25.34) and for (iv) ([4], p.18), while (v) is obvious.

We know now that all self-dual LCA group are $dn - groups$. However, an arbitrary $dn - group$ may or may not be self-dual.

EXAMPLE 2.3

(i) Let $G = Q \times T$ where Q is discrete group of rationals and T is the compact group of the unit circle. Then $\hat{G} = \hat{Q} \times Z$. Now G and $\hat{G}$ are not self-dual, but G and $\hat{G}$ are $dn - groups$ for $dim\, G = dim\, \hat{G} = 1$.

(ii) Let $G = R^2 \times J_p$, where J_p is the compact group of $p - adic$ integers. Then $\hat{G} = R^2 \times Z(p^\infty)$, so G is not self-dual but $dimG = dim\, \hat{G} = 2$, and G is a $dn - group$.

The structure theorem of LCA groups states that each LCA group G is a topological direct product $G = R^n \times M$, where $n \geq 0$ and M has a compact, open subgroup. Since $\hat{G} = R^n \times \widehat{M}$, and R^n is a self-dual $dn - group$, it is the other factor M which may or may not be a $dn - group$. In short, $G = R^n \times M$ is a $dn - group$ if M is a $dn - group$.

DEFINITION 2.4

We shall say that an LCA group is R – free if it has no copy of R as a topological direct summand; equivalently, if it has a compact open subgroup.

We now describe a useful property of R – free LCA group.

THEOREM 2.5

Let G be an R – free LCA group. Then G_o compact, $b(G)$ is open, G_o + t(G) is contained in $b(G)$ and $G/b(G)$ is discrete and torsion – free. If H is a closed subgroup of G containing G_o, then $dim\, G = dim\, H$. In-particular, $dim\, G = G_o = dim\, b(G)$

Proof.

Since G is R-free, G_o is compact. Also $(G) \leq b(G)$. hence, $G_o + t(G) \leq b(G)$. As $b(G) + G_o$ is open in general ([4], p.17), so $b(G)$ is an open subgroup of R – free G . Also $G/b(G)$, being dual of $(\hat{G})_o$, is torsion free. By ([6], theorem 33.5), $dim\, G_o = \dim G$. Since $G_o \leq H \leq G$, it is clear that $dim\, G = dim\, H$. The last assertion is now obvious.

Our next theorem is about a property of finite-dimensional $dn - groups$.

THEOREM 2.6

Let G be a R-free dn-group with $dim\, G = n < \infty$, and let H be a closed subgroup of G . Then the following are equivalent:

(i) $dim\, H = dim\, A(\widehat{G}, H) = n$,

(ii) $G_o \leq H \leq b(G)$ and $(\hat{G})_o \leq A(\widehat{G}, H) \leq b(\hat{G})$

Proof.

(i) $\Longrightarrow$ (ii) : $G_o \leq H$ and $(\hat{G})_o \leq A(\widehat{G}, H)$ by ([6], 33.5(b)). By duality,

$A(\widehat{G}, H) \leq A(\widehat{G}, G_o) = b(\widehat{G})$. Similarly, $H \leq b(G)$.

(ii) $\Longrightarrow$ (i) : $G_o \leq H$ implies $dim\, H = dim\, G_o = n$, and similarly for $dim\; A\,((G\,)\hat{}, H) = n$

Before proceeding further on R – groups, we recall basic facts on independence and rank of subsets of an abelian group ([7], Page 83), and modify them a bit to suit our purposes.

DEFINITION 2.7

A subset S of an abelian group A is said to be independent of every equality

$n_1 s_1 + n_2 s_2 + \cdots\cdots + n_k s_k = 0$ with $n_i \in Z$, $s_i \in S$ implies that

$n_1 = n_2 = \cdots\cdots = n_k = 0$.

The following properties follow:

(i) The subgroup $< S >$ generated by an independent set S is free.

(ii) Every independent set S is contained in a maximal one, say S_1 , where $A/< S_1 >$ is torsion. Any two maximal independent subsets of A have the same cardinality.

The cardinality $r(A)$ or rA of any maximal independent subset of A is called the free rank of A , and $r(A) = r(H) + r(A/H)$ for any subgroup H of A .

We now present a general theorem on $dn - groups$.

THEOREM 2.8

The following statements are equivalent for an $R - free$ LCA group G :

(a) G is a $dn - group$,

(b) $\dim G_o = dim(\,\widehat{G}\,)_o = \dim b\, G = \dim b\, \widehat{G}$,

(c) $r(G/bG) = \dim G$,

(d) $r(\widehat{G}/b\widehat{G}) = \dim \widehat{G}$.

Proof.

(a) $\Longrightarrow$ (b): $dim\, G = dim\, G_o = + \dim(G/G_o) = dim\, G_o$, because G/G_o is totally disconnected and 0 – $dimensional$. So $dim\, G = dim\, G_o = dim\; \widehat{G} = dim\; (\widehat{G})_o$ in the same way. Now $G = dim\, bG + \dim(G/bG)$, the last being discrete and $0 - dimensional$, we have $dim\, G = \dim bG = \dim G_o$. Similarly, $dim\; (\widehat{G})o = dim\; b\widehat{G} = dim\widehat{G}$.

(b) $\Longrightarrow$ (a): We have $dim\, G = dim\, G_o$ and $\dim \hat{G} = dim\, (\hat{G})_o$, so $dim\, G = \dim \hat{G}$.

(a)$\Longrightarrow$ (c): $dim\, G = dim\, \hat{G} = dim\, (\hat{G})_0$. Since $(\hat{G})_o$ is compact, $dim\, (\hat{G})_0 = free$ rank of the discrete torsion free group G/bG . It follows that $dim\, G = r(G/bG)$.

(c) $\Longrightarrow$ (a): Since $(\hat{G})_0$ is compact, $dim\, (\hat{G})_0 = free$ rank of G/bG. ([8], theorem 3.3.13). Hence $(\hat{G})_0 = dim\ G$, but $dim\, (\hat{G})_0 = dim\, \hat{G}$,so we get $dim\, G = dim\, \hat{G}$, G is a $dn - group$.

(d) $\Longrightarrow$ (a): The proof is exactly similar to the above (a)$\Leftrightarrow$(c), if we observe that

$dim\, b\hat{G} = \dim \hat{G}$ (theorem 2.5).

Using the results proved above, we shall find several classes o f $dn - groups$.

COROLLARY 2.9

The following statements are equivalent for a compact abelian group G:

(i) G is a $dn - group$,

(ii) G is totally disconnected,

(iii) G is $-dimensional$,

(iv) $\hat{G}$ is a discrete torsion group

Proof.

(i) $\Leftrightarrow$ (iii):Since G is compact, $bG = G$, so $r(G/bG) = 0$, so $dimG = 0$ (theorem 2.8).

The equivalence of (ii), (iii) and (iv) is known in general ([6], 23.29, 23.30).

COROLLARY 2.10

A connected LCA group G is a $dn - group$ iff $G \cong R^n$, for some $n \geq 1$.

Proof.

G connected is of the form $R^n \times M$, where M is compact and connected. But by Corollary 2.9, M must be trivial, and the Corollary follows.

We recall that an LCA group generated by a compact subset is called compactly generated and is of the form $R^m \times Z^n \times C$, where C is compact.

COROLLARY 2.11

A compactly generated LCA group $R^m \times Z^n \times C$ is a $dn - group$ iff $dim\ C = n$.

Proof.

We need consider the $R - free$ part $Z^n \times C$. Here $b(Z^n \times C) = C$, and $(Z^n \times C)/C \cong Z^n$, and $r(Z^n) = n$, so by theorem 2.8, $Z^n \times C$ is a $dn - group$ iff $dim\ C = n$.

We recall that an LCA group G is called monothetic if G contains a dense cyclic subgroup. Then G is monothetic implies it is either discrete group Z or is compact. Moreover, a compact abelian group G is monothetic if $\hat{G}$ is isomorphism to a subgroup of T_d, the discrete group of the unit circle group.

COROLLARY 2.12

A compact monothetic group G is a $dn - group$ iff $G \cong \Pi_{p \in P} A_p$, where for each prime $p \in P$, $A_p \cong J_p$ or $Z(p^n)$ for some non-negative integer n.

Proof.

Clearly G is a $dn - group$ iff $\hat{G}$ is a subgroup of Q/Z. From this and using duality, the results follows. Clearly, every abelian group becomes an LCA group with the discrete topology. But, in general, infinite abelian groups admit more than one locally compact group topology, and in many cases, infinite number of locally compact group topologies. ([4], Chap. 10). It follows in particular from theorem 10.1 [4] that if a torsion abelian group A has a non-discrete locally compact group topology, then it has finitely (sometimes infinitely) many more such stronger topologies. We shall show now that torsion LCA groups are more amenable to be $dn - groups$ than the rest of LCA groups.

For our last result, we introduce a notation. If A is a abelian group, let $T(A)$ denote the set of all topologies that turn A into an LCA group.

COROLLARY 2.13

An abelian group A is a $dn - group$ under all topologies $T \in T(A)$, iff A is a torsion group.

Proof.

First, let A be a torsion group and $T \in T(A)$. Then (A, T) is totally disconnected and so

$o-dimensional$. Also $(A,T)^{\wedge}$ is totally disconnected , for every homomorphic image of A is torsion. Thus (A,T) is a $dn-group$. Now suppose A is not a torsion group. Then A with the discrete topology has free rank ≥ 1, and so the compact group $\hat{A}$ has dimension ≥ 1 . It follows that in this case A is not a $dn-group$.

3. CONCLUSION

In this study, some conditions have been investigated such that the dimension of an arbitrary LCA group may equal the dimension of its dual LCA group. Following results with the help of examples were found:

(i) if $G = Q \times T$, then it is a $dn-groups$ for $dim\, G = dim\, \hat{G} = 1$.

(ii) if $G = R^2 \times J_p$, then G is not self-dual but is a $dn-group$.

(iii) If G = a R-free $dn-group$ with $dim\, G = n < \infty$, then the following statements are equivalent:

  a. $dim\, H = dim\, A(\widehat{G}, H) = n$,

  b. $G_o \leq H \leq b(G)$ and $(\widehat{G})_o \leq A(\widehat{G}, H) \leq b(\widehat{G})$

(iv) If $G = R-free$ LCA group, then the following statements are equivalent:

  a. G is a $dn-group$,

  b. $\dim G_o = dim(\widehat{G})_o = \dim b\, G = \dim b\, \widehat{G}$,

  c. $r(G/bG) = \dim G$,

  d. $r(\widehat{G}/b\widehat{G}) = \dim \widehat{G}$.

(v) $R^m \times Z^n \times C$ is a $dn-group$ iff $dim\, C = n$.

(vi) An abelian group A is a $dn-group$ under all topologies $T \in T(A)$, iff A is a torsion group.

4. REFERENCES

1. J-Nagata, Modern dimension theory, North Holland Publishing Company, 1965.
2. A. R. Pears, Dimension theory of general spaces. Cambridge University Press, New York, 1975.
3. B. A. Pasynkov, The coincidence of various definitions of dimensionality for locally compact groups, Soviet Math. 1, 1960.
4. D.L. Armacost, The structure of locally compact abelian groups, Marcel Dekker, New York, 1981.

5. E. Hewitt and K. Ross, Abstract Harmonic Analysis, Vol. 1, Springer-Verlag, 1963.
6. M. Stroppel. Locally compact groups. European Mathematical Society (EMS), Zürich, 2006.
7. László Fuchs, Infinite abelian groups. Vol. I, Academic Press, New York, 1970.
8. L.N. Stoyanov and D.N. Dikranjan, Topological Groups, Marcel Dakkar, New York, 1990.