Exact Edge Domination in Graphs

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Abstract- Let $G = (V, E)$ be a connected graph. Let $G = (V, E)$ be any connected graph. Let $X \subseteq E$. The set X is said to be an *exact edge dominating set* , if $|N(e_i) \cap X| = 1$ and $|N(e_j) \cap X| \leq 1$ for every $e_i \in E(G) - X$ and $e_j \in X$. An exact edge dominating set is denoted as ExED set. The exact edge domination number $\gamma'_e(G)$ of a graph equals the cardinality of a minimum exact edge dominating set.In this paper, the features of exact edge dominating sets in the given graphs are derived. Also the bounds of size and diameter of the graphs are defined in terms of maximum degree $\Delta(G)$. We prove that in a connected graph G with $\gamma'_e(G) = l$. Then $2l \leq m \leq 2l(\Delta(G) + 1)$.

Index Terms- Exact dominating set, exact edge dominating set, wounded spider, corona graph

I. INTRODUCTION

For standard notations we do not introduce here, the reader is always referred to the introductory chapter of [3]. Domination in graphs has been studied extensively in recent years. The book by Haynes, Hedetniemi, and Slater [4] is entirely devoted to this area.

Let $G = (V, E)$ be a simple , finite , connected and undirected graph. The exact domination in graphs concept was introduced by Anto Kinsley[1]. The order and size of G are dentoed by n and m respectively. For basic graph theoretic terminology we refer to G. Chartrand [3]. A set of vertices $S \subseteq V$ is called a *dominating set* of G if every vertex of $G$is dominated by at least one member of S. Equivalently a dominating set is efficient if the distance between any two vertices in S is at least three, that is S is a packing. Two edges in a graph are independent if they are not adjacent in G. A set of pairwise independent edges of G is called a *matching* in G. While a matching of maximum cardinality is a maximum matching. If M is a matching in a graph G with the propertythat every vertex of G is incident with an edge of M, then M is a perfect matching in G. Clearly if G has a perfect matching M, then G has even order and $<M>$ is a 1-regular spanning subgraph of G.The *corona* of two graphs G_1, and G_2 is the graph $G = G_1 \odot G_2$formed from one copy of G_1 and $|V(G_1)|$ number of copies of G_2 where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2 for $1 \leq i \leq |v(G_1)|$. A graph G is said to be a wounded spider formed by subdividing at most $t - 1$ of the edges of a star $K_{1,t}$ for $t \geq 0$. The concept of edge domination was introduced by Mitchell and Hetetniemi[5]. The required basic definitions are studied from Haynes T. W, et all. [6]. This paper is fascinated on exact edge domination in graphs. Throughout this paper, P_n, C_n, and $K_n$will stand for the path, cycle and complete graph with order n respectively.

II. EXACT EDGE DOMINATING SET

Definition 2.1

Let $G = (V, E)$ be any connected graph. Let $X \subseteq E$. The set$X$ is said to be an *exact edge dominating set* , if $|N(e_i) \cap X| = 1$ and $|N(e_j) \cap X| \leq 1$ for every $e_i \in E(G) - X$ and $e_j \in X$. An exact edge dominating set is denoted as ExED set.

Definition 2.2

The exact edge domination number $\gamma'_e(G)$ of a graph equals the cardinality of a minimum exact edge dominating set.

Example 2.3

Consider the graph G,

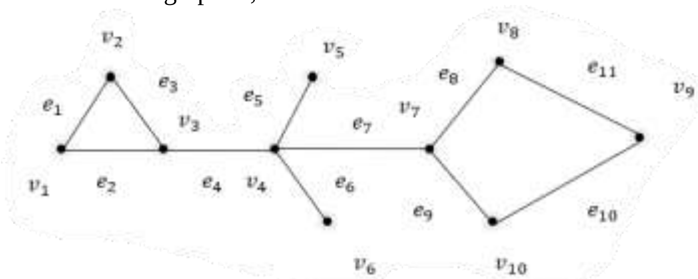

Figure 2.1: *A graph G for exact edge dominating set*

In Figure [2.1], The set $X = \{e_1, e_5, e_{10}, e_{11}\}$ forms ExED set. Also the set $\{e_1, e_6, e_{10}, e_{11}\}$ is an ExED set. But the set $\{e_2, e_7, e_{11}\}$ is a edge dominating, but not a ExED set.

The parameter $\gamma'_e(G)$ cannot be computed for some graphs. For example, cycle C_5 not having ExED set.

Example 2.4

Consider the graph G'

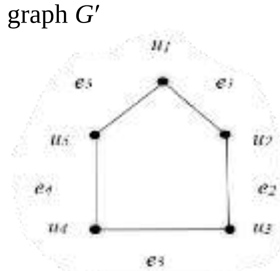

Figure 2.2. *A graph G' not having ExED set*

In Figure [2.2], Let $X' = \{e_1, e_2\}$, then $|N(e_4) \cap X'| = 0$. Then X' is not an ExED set. Suppose $X' = \{e_1, e_2, e_4\}$, then $N(e_5) \cap X' = \{e_1, e_4\}$ and $N(e_3) \cap X' = \{e_2, e_4\}$. That is $|N(e_5) \cap X'| = |N(e_3) \cap X'| = 2 \neq 1$ where $e_5, e_3 \in V - X'$. Then X' is not an ExED set.

In figure 1, Add an edge $e_{12} = v_7v_9$ in G, then G has no ExED set.

Theorem 2.5

Let G be any connected graph and with the condition $deg(e_i) = 1$, $deg(e_j) > 1$ and $e_i$and $e_j$are adjacent edges in G. If X is an ExED set and $e_i, e_j \in X$, then X is not aExED set.

Proof

Let X be an ExED set and $e_i, e_j \in X$. Then we have to prove that X is not a minimum ExED set, then either $X - \{e_i\}$ or $X - \{e_j\}$ is not an ExED set. The set $X - \{e_j\}$ is not an ExED set, since $\deg(e_j) > 1$ and by definition of ExED set the edges of the set $N(e_j) - \{e_i\}$ are not dominated by any other edges in X. In this case X is not a minimum ExED set. But the set $X - \{e_i\}$ is an ExED set. That is all the edges of $\{E(G) \cup \{e_i\}\} - X$ are dominated by any other edges in X.

Theorem 2.6

Let G be any connected graph order $n \geq 4$ and X be a minimum ExED set with $|N(e_i) \cap X| = 1$ for all $e_i \in X$, then $deg(e_i) \neq 1$.

Proof

Suppose $deg(e_i) = 1$ and $N(e_i) \cap X = \{e_j\}$ for $i \neq j$ where $e_i, e_j \in X$. Since $deg(e_i) = 1$, assume that u and v be two vertices incident with the edge e_i, then $deg(u) + deg(v) - 2 = 1$ which implies that,$deg(u) + deg(v) = 3$. Then either $deg(u) = 2$, $deg(v) = 1$ or $deg(u) = 1$, $deg(v) = 2$. Take $deg(u) = 1$, $deg(v) = 2$, which means that v is the support vertex of the vertex u. let $w \in N(V)$, then $e_i = uv$ and $e_j = vw$. By above theorem $X - \{e_i\}$ is an ExED set. Then X is not a minimum ExED set in G. Hence $deg(e_i) \neq 1$.

Remark 2.7

Let X be a ExED set with $N(e_i) \cap X = \{e_j\}$ where$e_i, e_j \in X$ and $u$be a vertex incident with both e_i and e_j. Then $deg(u) = 2$.

Remark 2.8

By the abovetheorems[2.5], [2.6], If S is a minimum ExED set, with $deg(e_i) = 1$, for all $e_i \in X$, then $|N(e_i) \cap X| = 0$.

Theorem 2.9

Let G be any connected graph and X be a ExED set in G. Let X be the set defined as the number of vertices incident with the edges in X. If $|N(e_i) \cap X| = 0$for all $e_i \in X$, then X contains even number of vertices.

Proof

By our assumption, $N(e_i) \cap X = \emptyset$, for all $e_i \in X$. Every edge is incident with two vertices. Let $|X| = k$. The k edges are incident with $2k$ vertices. Hence X contains even number of vertices.

Theorem 2.10

If X is a ExED set in $G \odot H$ with $\gamma'_e(G \odot H) = 1$ if and only if $G \cong K_2$ and $H \cong K_1$ or $G \cong K_1$ and $H \cong K_2$.

Remark 2.11

Let G and H be a connected graph of order n_1 and n_2 respectivley. Suppose $n_1 > 2$ or $n_2 > 2$, then $G \odot H$ has no ExED set.

Theorem 2.12

Let G wounded spider graph. Then $\gamma'_e(G) = 2$, for $s = 1$ and for $2 \leq s \leq t - 1$, G does not have an ExED set.

Theorem 2.13

The Complete graph K_n, $n > 3$, has no ExED set.

Proof

Suppose X be an ExED set in K_n. Suppose $|X| = 2$ and $e_1, e_2 \in X$. Then by definition $|N(e_1) \cap X| \leq 1$ and $|N(e_2) \cap X| \leq 1$, for $e_1, e_2 \in X$. Suppose $|N(e_1) \cap X| = 1$, then obviously $N(e_1) \cap X = \{e_2\}$. Take $a_1a_2 = e_1$ and $a_2a_3 = e_2$. But for $n \geq 4$, $deg(a_2) \geq 3$, by theorem [2.6], remark[2.7], is a contradiction. Suppose $|N(e_1) \cap X| = 0$ and $|N(e_2) \cap X| = 0$. Take $a_1a_2 = e_1$ and $a_3a_4 = e_2$. In K_n, a_3 is adjacent to a_1 and a_2, similarly a_4 is adjacent to a_1 and a_2. Then there exits an edge e_l such that $|N(e_l) \cap X| = 2$, for $e_l \in E(G) - X$, which is a contradicts our assumption that X is an ExED set. For $|X| > 2$, we get the above similar cases. Hence we can conclude that, K_n has no ExED set.

Remark 2.14

For K_n, $n = 3$, then $\gamma'_e(K_n) = 1$.

Remark 2.15

For K_n, $n \leq 2$, then K_n has no ExED set.

Theorem 2.16

The Wheel graph W_n, $n \geq 4$ has no ExED set.

Theorem 2.17

Let X be an ExED in G with $\gamma'_e(G) = l$ and $X'_e(G) = \{x, y \in V(G) / xy = e_i$, for all $e_i \in X$ where $1 \leq i \leq l\}$. Then we have the following:

(i). when l is even, $\langle X'_e(G) \rangle = \begin{cases} lP_2 \\ \left(\frac{l}{2}\right) P_3 \\ (2s)P_2 \cup \left(\frac{l-2s}{2}\right) P_3 \end{cases}$, where $1 \leq s \leq \left(\frac{l-2}{2}\right)$

(ii). when l is odd $\langle X'_e(G) \rangle = \begin{cases} lP_2 \\ (2t+1)P_2 \cup \left(\frac{l-(2t+1)}{2}\right) P_3 \end{cases}$, where $0 \leq t \leq \left(\frac{l-3}{2}\right)$.

Remark 2.18

By the theorem[2.17] we have $\frac{3l}{2} \leq \left|X'_e(G)\right| \leq 2l$, when l is even and $\frac{3l+1}{2} \leq \left|X'_e(G)\right| \leq 2l$, when l is odd.

When l is even, for the upper bound of $\left|X'_e(G)\right|$, we have $\langle X'_e(G) \rangle = lP_2$. Then $\left|X'_e(G)\right| = 2l$. And for lower bound of $\left|X'_e(G)\right|$ occurs when $\langle X'_e(G) \rangle = \left(\frac{l}{2}\right) P_3$. Then $\left|X'_e(G)\right| = \frac{3l}{2}$.

Consider $\langle X'_e(G) \rangle = (2s)P_2 \cup \left(\frac{l-2s}{2}\right) P_3$, where $1 \leq s \leq \left(\frac{l-2}{2}\right)$.

When $s = 1$, then $\left|X'_e(G)\right| = (2 \times 1)2 + 3\left(\frac{l-1}{2}\right) = \frac{8+3l-6}{2} = \frac{3l+2}{2}$. When $s = \frac{l-2}{2}$, then $\left|X'_e(G)\right| = \left(2\left(\frac{l-2}{2}\right)\right)2 + 3\left(\frac{l-2\left(\frac{l-2}{2}\right)}{2}\right) = 2l - 4 + 3 = 2l - 1$.Therefore $\frac{3l+2}{2} \leq \left|X'_e(G)\right| \leq 2l - 1$, for $1 \leq s \leq \left(\frac{l-2}{2}\right)$.

Similarly, when l is odd, the upper bound of $|X_e'(G)|$occurs, when $\langle X_e'(G)\rangle = lP_2$.And for the lower bound of $|X_e'(G)|$, consider $\langle X_e'(G)\rangle = (2t+1)P_2 \cup \left(\frac{l-(2t+1)}{2}\right)P_3$, where $0 \le t \le \left(\frac{l-3}{2}\right)$.When $t = 0$, $|X_e'(G)| = 2 + \left(\frac{l-1}{2}\right)3 = \frac{4+(l-1)3}{2} = \frac{3l+1}{2}$. When $t > 0$, we have $|X_e'(G)| > \frac{3l+1}{2}$. Therefore, $\frac{3l+1}{2} \le |X_e'(G)| \le 2l$.

III. BOUNDS ON SIZE AND DIAMETER OF THE GRAPH G WITH RESPECT TO MAXIMUM DEGREE IN G

Theorem 3.1

Let m be the size and $\Delta(G)$ be the maximum degree in G with $\gamma_e'(G) = l$. Then $2l \le m \le 2l(\Delta(G)+1)$.

Proof

Let X be an ExED set in G with $\gamma_e'(G) = l$. Let $X = \{e_{l_1}, e_{l_2},\ldots,e_{l_l}\}$ be an ExED set and $S_X = \{a_1, a_2, \ldots, a_{2l}\}$ be the set of vertices incident with the edges of X. For upper bound of m, consider $|N(e_j) \cap X| = 0$, for all $e_j \in X$. Then $deg(a_i) \le \Delta(G)$, for $1 < i \le 2l$. Suppose $deg(a_i) = \Delta(G)$, for all a_i, then $\Delta(G)$ number of vertices incident with each a_i. Take $a_ib_{i_j} = e_{i_j}$, where $e_{i_j} \in E(G) - X$ and $1 \le j \le \Delta(G)$. Since G is connected, then $X_i = \{e_{i_j}/a_ib_{i_j} = e_{i_j}$, where $1 \le j \le \Delta(G)\}$ for $1 < i \le 2l$ is the set which consists the edges in $E(G) - X$. Then $m = |X| + |X_i| = l + \underbrace{\Delta(G) + \Delta(G) + \cdots \ldots + \Delta(G)}_{2l\ times} = l + 2l\Delta(G) = l(1+2\Delta(G))$. Suppose $deg(a_i) < \Delta(G)$, then $m < l(1+2\Delta(G))$. Hence $m \le 2l(\Delta(G)+1)$for $deg(a_i) \le \Delta(G)$.

For lower bound of m, consider $|N(e_{l_a}) \cap X| \le 1$, for all $e_{l_a} \in X$. Then we have following two cases.

Case (i). When l is odd with $N(e_{l_k}) \cap X = \{e_{l_{k+1}}\}$, where $k = 2r + 1$, for $r = 0, 1, 2, \ldots, \frac{l-3}{2}$ and $N(e_{l_l}) \cap X = \emptyset$ with respect to X. Take $deg(a_i) = 2$, since G is connected, $m = 4\left[\left(\frac{l-3}{2}\right) + 1\right] + 3 = 2(l-1) + 3 = 2l + 1$. When $deg(a_i) > 2$, we get $m > 2l + 1$. Then we can conclude that $m \ge 2l + 1$, when l is odd with $|N(e_{l_a}) \cap X| \le 1$, for all $e_{l_a} \in X$.

Case (ii). When l is even with $deg(a_i) = 2$ with $|N(e_{l_a}) \cap X| = 1$, then by the above case we have $m = \frac{4l}{2} = 2l$. When $deg(a_i) > 2$, we get $m > 2l$. Therefore, $deg(a_i) \ge 2$, we get $m \ge 2l$. Hence by above all the case, $2l \le m \le 2l(\Delta(G)+1)$.

Theorem 3.2

Let G be a connected graph with $\gamma_e'(G) = l$, then $diam(G) \le 3l$.

Proof

Let X be an ExED set in G with $\gamma_e'(G) = l$. By defintion of an ExED set we have $|N(e_i) \cap X| = 1$ and $|N(e_j) \cap X| \le 1$ for every $e_i \in E(G) - X$ and $e_j \in X$. Let us now consider the case $|N(e_j) \cap X| = 0$ for all $e_j \in X$. Let $S_X = \{u_1, u_2, u_3, \ldots, u_{2l}\}$ be the set of vertices which are incident with edges of X. For upper bound of $diam(G)$, let us now consider the diametrical path d which consists of all the l number of edges of X. Then $e(u_a) \le 3l - 2$, for all $u_a \in S_X$and $e(u_b) \le 3l$, for all $u_b \in V(G) - S_X$. Therefore, $max\{e(u_x)\} = 3l$, which means that $diam(G) = 3l$, for every $u_x \in V(G)$. Suppose that $|N(e_j) \cap X| \le 1$, for all $e_j \in X$. Then for lower bound of diameter of G, we have the following two cases.

Case(i). When l is even with $|N(e_j) \cap X| = 1$, for all $e_j \in X$, then $e(u_a) \le 2l - 1$, for all $u_a \in S_X$and $e(u_b) \le 2l$, for all $u_b \in V(G) - S_X$. Therefore, $max\{e(u_x)\} = 2l$, which means that $diam(G) = 2l$, for every $u_x \in V(G)$ in this case.

Case(ii). When l is odd with $|N(e_j) \cap X| \le 1$, for all $e_j \in X$, then $e(u_a) \le 2l$, for all $u_a \in S_X$and $e(u_b) \le 2l + 1$, for all $u_b \in V(G) - S_X$. Therefore in this case, $max\{e(u_x)\} = 2l + 1$, which means that $diam(G) = 2l + 1$, for every $u_x \in V(G)$.

From all the above cases $max\{e(u_x)\} = 3l$, for $u_x \in V(G)$, that is $diam(G) = 3l$, for $u_x \in V(G)$ with $|N(e_j) \cap X| = 0$ for all $e_j \in X$. If atmost $l - 1$ number of edges lie on the diametrical path d, then $diam(G) < 3l$. Therefore, we can conclude that $diam(G) \le 3l$, for every $u_x \in V(G)$.

Theorem 3.3

Let X be a ExED set in a connected graph G with $\gamma_e'(G) = l$, and l is even where $l \ge 4$ and $\Delta(G)$ be the maximum degree of G, then $diam(G) \ge 8$, for $\Delta(G) \ge \frac{l}{2}$ and $diam(G) \ge 10$, for $\Delta(G) < \frac{l}{2}$.

Proof

Let X be a ExED set in G with $\gamma_e'(G) = l$ and $\Delta(G)$ be the maximum degree of G. For lower bound of diameter of G, let us consider $|N(e_{j_1}) \cap X| = 1$, for all $e_{j_1} \in X$. Take $N(e_{j_1}) \cap X = \{e_{i_1}\}$ for $e_{i_1} \in X$. let u_a be a vertex in G such that $u_au_{b_1} = e_{l_1}$, such that $deg(u_a) = \Delta(G)$; where $e_{l_1}$is the edge incident with the vertices u_{b_1} and u_a and $u_{b_1}$is the vertex $u_{b_1}u_{g_1} = e_{i_1}$. Then we have following two claims for getting the lower bound of diameter of G.

Claim A. Suppose $\Delta(G) = \frac{l}{2}$, Since X is an ExED set in G, then there exits an edges e_{i_f}, e_{j_f} such that $N(e_{j_f}) \cap X = \{e_{i_f}\}$, for $2 \le f \le \Delta(G)$. Also by defintion of and ExED-set , the edges e_{j_x} are adjacent to the edges e_{w_x} , where $u_{d_x}u_{c_x} = e_{w_x}$, for $1 \le x \le \Delta(G)$.

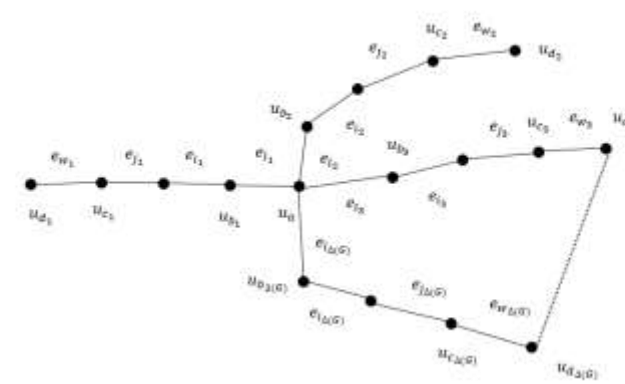

Figure 3.1. The graph G having diam(G) ≥ 8, for Δ(G) ≥ $\frac{1}{2}$

From Figure[3.1] we can easily say that $e(u_{d_y}) = d(u_{d_y}, u_a) + d(u_a, u_{d_z}) = 4 + 4 = 8$, for all $u_{d_z}, u_{d_y} \in V(G)$ with $z \ne y$ and $1 \le y, z \le \Delta(G)$, which is the maximum eccentricity in G. Then

$diam(G) = 8$. Suppose that $deg(u_a) < \Delta(G)$, then $diam(G) > 8$. Hence we can conclude that $diam(G) \geq 8$, for $\Delta(G) = \frac{l}{2}$ with $l \geq 4$.

Claim B. Assume that, $\Delta(G) < \frac{l}{2}$, then $\Delta(G) + \xi = \frac{l}{2} \Rightarrow l = 2(\Delta(G) + \xi)$ where $1 \leq \xi \leq \frac{l-2\Delta(G)}{2}$. Suppose $deg(u_a) = \Delta(G)$ and $deg(u_{b_x}) = \Delta(G)$ with $l = 2(\Delta(G) + \xi)$, where $1 \leq \xi \leq \Delta(G)(\Delta(G) - 2)$, by the above case, for lower bound of diameter of G, there exists atmost $\Delta(G) - 2$ number of edges adjacent to each u_{b_x}, where $1 \leq x \leq \Delta(G)$.

Let u_{ax_s} be the set of vertices adjacent to u_{b_x}, where $1 \leq s \leq \Delta(G) - 2$. Take $u_{bx_s}u_{ax_s} = e_{lx_s}$ such that $N(e_{lx_s}) \cap X = \{e_{ix_s}\}$, where $e_{lx_s} \in E(G) - X$ and $e_{ix_s} \in X$. By our assumption $N(e_{ix_s}) \cap X = \{e_{jx_s}\}$, where $e_{jx_s} \in X$, where $1 \leq x \leq \Delta(G)$ and $1 \leq s \leq \Delta(G) - 2$. By definition of ExED set in G, there exists edges $e_{wx_s} = u_{cx_s}u_{dx_s}$, such that $N(e_{wx_s}) \cap X = \{e_{jx_s}\}$.

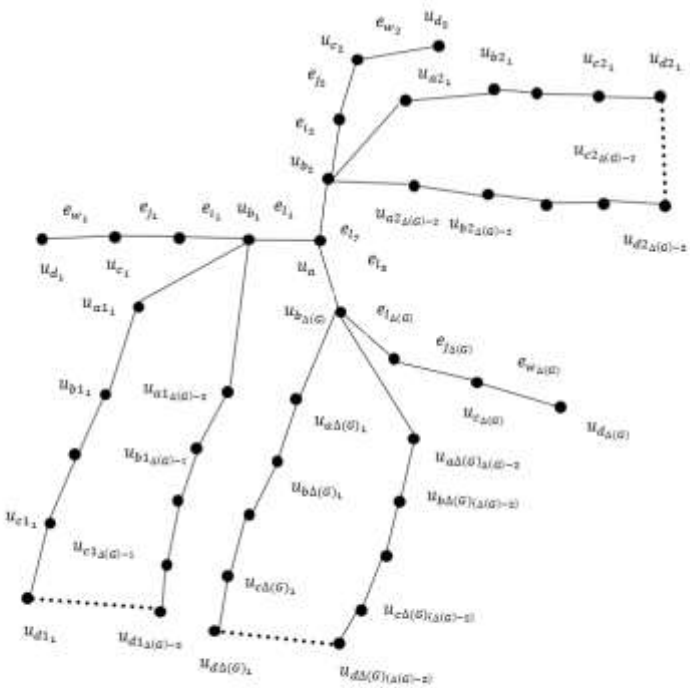

Figure 3.2. The graph G having diam(G) = 10 when $\Delta(G) < \frac{l}{2}$ i.e., $l = 2(\Delta(G) + \xi)$, where $1 \leq \xi \leq \Delta(G)(\Delta(G) - 2)$)

From figure[3.2], $e(u_{dx_g}) = d(u_{dx_g}, u_a) + d(u_a, u_{dr_h}) = 4 + 6 = 10$, where $1 \leq g, h \leq (\Delta(G) - 2)$ and $1 \leq x, r \leq \Delta(G)$ with $h \neq g$ and $r \neq x$, which gives the maximum eccentricity in G. Then $diam(G) = 10$, for $1 \leq \xi \leq \Delta(G)(\Delta(G) - 2)$. Suppose $deg(u_a) < \Delta(G)$ and $deg(u_{b_x}) < \Delta(G)$ with $l = 2(\Delta(G) + \xi)$, where $(\Delta(G) - 1)^2 \leq \xi \leq \frac{l-2\Delta(G)}{2}$, then $diam(G) > 10$. Therefore $diam(G) \geq 10$, for $\Delta(G) < \frac{l}{2}$.

IV. CONCLUSION

This paper discusses and analyses the exact edge domination number for some standard graph. Using the exact edge domination number the diameter and size of the graph are disclosed.

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