Centered Hexagonal Graceful Labeling of Caterpillar and Uniform Caterpillar Graphs

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ABSTRACT

The n^{th} centered hexagonal number is denoted by D_n and is of the form $D_n = 3n(n-1)+1$. Let G be a (p,q) graph. Let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G respectively. A centered hexagonal graceful labeling of a graph G is an one to one function $f : V(G) \to \{0,1,2,...,D_q\}$ that induces a bijection $f^* : E(G) \to \{D_1, D_2, ..., D_q\}$ of the edges of G defined by $f^*(e) = |f(u) - f(v)|, \forall e = uv \in E(G)$. The graph which admits such a labeling is called a centered hexagonal graceful graph. In this paper, centered hexagonal graceful labeling of caterpillar and uniform caterpillar graphs are studied.

Keywords: centered hexagonal numbers, centered hexagonal labeling, centered hexagonal graceful graph.

1. INTRODUCTION

The graphs considered in this paper are finite, undirected and without loops or multiple edges. Let $G = (V, E)$ be a graph with p vertices and q edges. Terms not defined here are used in the sense of Harary [2]. For number theoretic terminology [1] is followed.

A graph labeling is an assignment of integers to the vertices or the edges or both subject to certain conditions. If the domain of the mapping is the set of vertices (edges/both) then the labeling is called a vertex (edge/ total) labeling.

There are several types of graph labeling and a detailed survey is found in [3]. In 1967, Rosa [6] presented four hierarchically related labeling of graphs, which he named α, β, σ and ρ valuations. In 1972, β- valuation had been called graceful labeling by Golomb [4]. Ramesh and Syed Ali Nisaya [5] introduced some more polygonal graceful labeling of path. For more information related to graph labeling, see [7,8].

2. PRELIMINARIES

Definition 2.1: A **path** P_n is obtained by joining u_i to the consecutive vertices u_{i+1} for $1 \le i \le n-1$.

Definition 2.2: Let $v_1, v_2, ..., v_n$ be the n vertices of the path P_n. From each vertex v_i, $i = 1,2,..., n$ there are m_i , $i = 1,2,..., n$ pendant vertices say $v_{i1}, v_{i2}, ..., v_{im_i}$. The resultant graph is a **caterpillar** and is denoted by $B(m_1, m_2, ..., m_n)$. The graph $B(m_1, m_2)$ is called a bistar graph. The caterpillar graph can also be defined in the following way.

G is called a caterpillar if G is a tree such that the removal of the vertices with degree 1 results in a path and that path is called the spine of the caterpillar.

Definition 2.3: A uniform caterpillar is a caterpillar with each vertex is either of degree 1 or of degree m where $m = \Delta(G)$. We denote a uniform caterpillar with n vertices on the spine by $Cat_{n,m}$.

Definition 2.4: A **graceful labeling** of a graph G is an one to one function $f : V(G) \to \{0, 1, 2, ..., q\}$ that induces a bijection $f^* : E(G) \to \{1, 2, ..., q\}$ of the edges of G defined by $f^*(e) = |f(u) - f(v)|$, $\forall\, e = uv \in E(G)$. The graph which admits such a labeling is called a **graceful graph**.

Definition 2.5: A centered hexagonal number is a centered figurate number that represents a hexagon with a dot in the center and all other dots surrounding the center in successive hexagonal layers. The n^{th} centered hexagonal number is $D_n = n^3 - (n-1)^3 = 3n(n-1)+1$. The first few centered hexagonal numbers are 1,7,19,37,61,91,127,169,217,271,331,397 etc.

Definition 2.6: A centered hexagonal graceful labeling of a graph G is an one to one function $f: V(G) \rightarrow \{0,1,2,...,D_q\}$ that induces a bijection $f^*: E(G) \rightarrow \{D_1, D_2, ..., D_q\}$ of the edges of G defined by $f^*(e) = |f(u) - f(v)|$, $\forall\, e = uv \in E(G)$. The graph which admits such a labeling is called a **centered hexagonal graceful** graph.

3. MAIN RESULTS

Theorem 3.1: The caterpillar $B(1,2,3,...,n)$ is a centered hexagonal graceful graph.

Proof: The caterpillar $B(1,2,3,...,n)$ is obtained from a path P_n by attaching i^{th} vertex of P_n with i pendant vertices. Let $G = (V, E)$ be the caterpillar $B(1,2,3,...,n)$. Let $V(G) = \{v_i, v_{ij} : 1 \le i \le n,\ 1 \le j \le i\}$ and $E(G) = \{v_{i-1}v_i : 2 \le i \le n\} \cup \{v_i v_{ij},\ 1 \le i \le n; 1 \le j \le i\}$

Here the path P_n has n vertices, $n-1$ edges and we are attaching $1+2+3+...+n = m$ (say) pendant vertices. Hence G has $m+n$ vertices $m+n-1$ edges.

Define $f: V(G) \rightarrow \{0,1,2,...,D_{m+n-1}\}$ as follows.

$$f(v_{2i-1}) = 6(i-1)(n-i), \text{ where } i = 1,2,...,\left(\frac{n+1}{2}\right)$$

$$f(v_{2i}) = \frac{1}{2}\left(6n^2 - 6n(1+2i) + 2(6i^2+1)\right), \text{ where } i = 1,2,...,\left(\frac{n}{2}\right) \text{ and}$$

$$f(v_{11}) = 3(m+n-1)(m+n-2)+1 \text{ and}$$

For $2 \le i \le n$, $f(v_{ij}) = f(v_i) + 3\left(m+n-j-\left(\frac{i^2-i}{2}\right)\right)\left(m+n-j-\left(\frac{i^2-i}{2}\right)-1\right)+1$, where $1 \le j \le i$.

We shall prove that G admits centered hexagonal graceful labeling. From the definition, it is clear that $\max_{v \in V(G)} f(v)$ is D_{m+n-1} and also $f(v) \in \{0,1,2,...,D_{m+n-1}\}$. Also from the definition, all the vertices of G have different labeling. Hence f is one to one. It remains to show that the edge values are of the form $\{D_1, D_2, ..., D_{n-1}, D_n ,..., D_{m+n-1}\}$. The induced edge function $f^*: E(G) \to \{D_1, D_2, ..., D_{n-1},\ D_n ,..., D_{m+n-1}\}$ is defined as follows.

$f^*(v_1 v_{11}) = 3(m+n-1)(m+n-2)+1$

For $2 \le i \le n$, $f^*(v_i v_{i-1}) = 3(n^2 + i^2 - 2in + n - i) + 1$

For $2 \le i \le n$, $f^*(v_i v_{ij}) = 3\left(m+n-j-\left(\frac{i^2-i}{2}\right)\right)\left(m+n-j-\left(\frac{i^2-i}{2}\right)-1\right)+1$, where $1 \le j \le i$.

Clearly f^* is a bijection and $f^*(E(G)) = \{D_1, D_2, ..., D_{n-1}, D_n ,.., D_{m+n-1}\}$. Therefore G admits centered hexagonal graceful labeling. Hence the graph G is a centered hexagonal graceful graph.

Example 3.2: The centered hexagonal graceful labeling of $B(1,2,3,4)$ is given in figure 1.

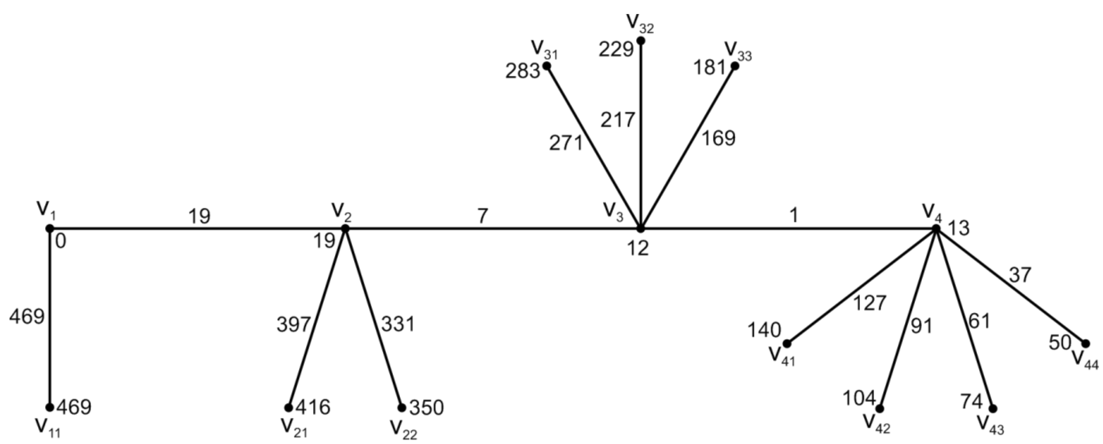

Figure 1

Now we prove that the uniform caterpillar $Cat_{n,m}$ for $n \ge 2$ is a centered hexagonal graceful graph. First we prove the following three lemmas.

Lemma 3.3: The uniform caterpillar $Cat_{2,m}$ is a centered hexagonal graceful graph.

Proof: Let $G=(V,E)$ be the uniform caterpillar $Cat_{2,m}$ with the vertex set $V=\{u_{10},u_{11},u_{12},...,u_{1m},u_{20},u_{21},u_{22},...,u_{2m}\}$ and the edge set $E=\{u_{10}u_{1j}\ :1\le j\le m\}\cup\{u_{20}u_{2j}\ :1\le j\le m\}\cup\{u_{10}u_{20}\}$. Then G has $2m+2$ vertices and $2m+1$ edges. Define $f:V(G)\to\{0,1,2,...,D_{2m+1}\}$ as follows.

$f(u_{10})=0$

$f(u_{20})=1$

$f(u_{ij})=f(u_{i0})+3[m^2(i^2-6i+9)+j^2-3m(i+2j-3)-3j+2ijm]+7$,where $i=1,2$ and $1\le j\le m$. We shall prove that G admits centered hexagonal graceful labeling. From the definition, it is clear that $\max\limits_{v\in V(G)} f(v)$ is D_{2m+1} and also $f(v)\in\{0,1,2,...,D_{2m+1}\}$. Also from the definition, all the vertices of G have different labeling. Hence f is one to one. It remains to show that the edge values are of the form $\{D_1,D_2,...,D_{2m+1}\}$. The induced edge function $f^*: E(G)\to\{D_1,D_2,...,D_{2m+1}\}$ is defined as follows.

$f^*(u_{10}u_{20})=1$

$f^*(u_{10}u_{1j})=3[4m(m-j)+j^2+3(2m-j)]+7$, where $1\le j\le m$

$f^*(u_{20}u_{2j})=3[m(m-2j)+3(m-j)+j^2]+7$, where $1\le j\le m$

Clearly f^* is one to one and $f^*(E(G))=\{D_1,D_2,...,D_{2m+1}\}$. Therefore G admits centered hexagonal graceful labeling. Hence the uniform caterpillar $Cat_{2,m}$ is a centered hexagonal graceful graph.

Example 3.4: The centered hexagonal graceful labeling of the uniform caterpillar $Cat_{2,4}$ is given in figure 2.

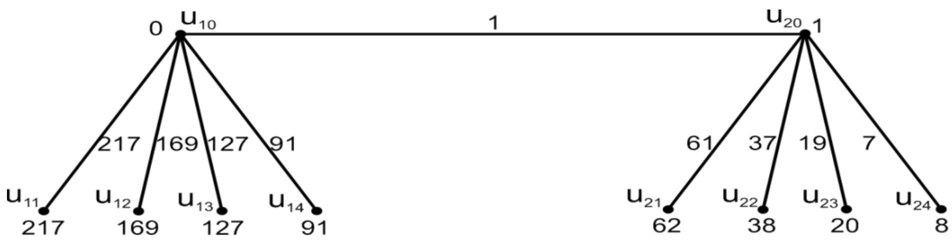

Figure 2

Lemma 3.5: The uniform caterpillar $Cat_{3,m}$ is a centered hexagonal graceful graph.

Proof: Let $G=(V,E)$ be the uniform caterpillar $Cat_{3,m}$ with the vertex set $V=\{u_{10},u_{11},u_{12},...,u_{1m},u_{20},u_{21},u_{22},...,u_{2m},u_{30},u_{31},u_{32},...,u_{3m}\}$ and the edge set $E=\{u_{10}\,u_{1j}\,,u_{20}\,u_{2j}\,,u_{30}\,u_{3j}\ :1\le j\le m\}\cup\{u_{10}u_{20},u_{20}u_{30}\}$. Then G has $3m+3$ vertices and $3m+2$ edges. Define $f:V(G)\to\{0,1,2,...,D_{3m+2}\}$ as follows.

$f(u_{10})=0$

$f(u_{i0})=f(u_{(i-1)0})+3i(i-3)+7$, where $i=2,3$

$f(u_{ij})=f(u_{i0})+3[m^2(i^2-8i+16)-m(5i+8j-20)+j^2-5j+2ijm]+19$, where $i=1,2,3$ and $1\le j\le m$

We shall prove that G admits centered hexagonal graceful labeling. From the definition, it is clear that $\max_{v\in V(G)} f(v)$ is D_{3m+2} and also $f(v)\in\{0,1,2,...,D_{3m+2}\}$. Also from the definition, all the vertices of G have different labeling. Hence f is one to one. It remains to show that the edge values are of the form $\{D_1,D_2,...,D_{3m+2}\}$. The induced edge function $f^*:E(G)\to\{D_1,D_2,...,D_{3m+2}\}$ is defined as follows.

$f^*(u_{10}\,u_{20})=1$

$f^*(u_{20}\,u_{30})=7$

$f^*(u_{10}\,u_{1j})=3[3m(3m-2j)+j^2+5(3m-j)]+19$, where $1\le j\le m$

$f^*(u_{20}u_{2j}) = 3[4m(m-j) + j^2 + 5(2m-j)] + 19$, where $1 \le j \le m$

$f^*(u_{30}u_{3j}) = 3[m(m-2j) + j^2 + 5(m-j)] + 19$, where $1 \le j \le m$

Clearly f^* is a bijection and $f^*(E(G)) = \{D_1, D_2, ..., D_{3m+2}\}$. Therefore G admits centered hexagonal graceful labeling. Hence the uniform caterpillar $Cat_{3,m}$ is a centered hexagonal graceful graph.

Example 3.6: The centered hexagonal graceful labeling of the uniform caterpillar $Cat_{3,4}$ is given in figure 3.

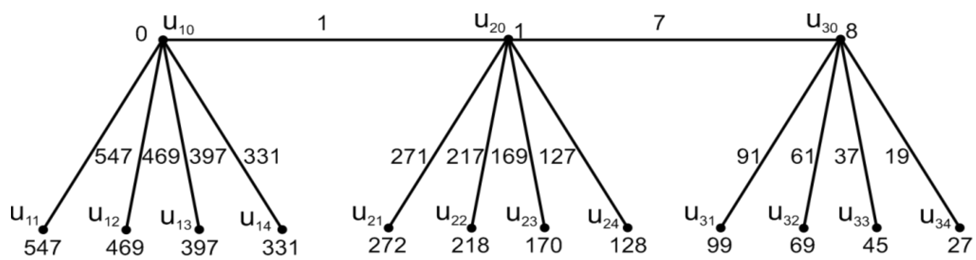

Figure 3

Lemma 3.7: The uniform caterpillar $Cat_{4,m}$ is a centered hexagonal graceful graph.

Proof: Let $G = (V, E)$ be the uniform caterpillar $Cat_{4,m}$ with the vertex set $V = \{u_{1j}, u_{2j}, u_{3j}, u_{4j} : 0 \le j \le m\}$ and the edge set $E = \{u_{10}u_{1j}, u_{20}u_{2j}, u_{30}u_{3j}, u_{40}u_{4j} : 1 \le j \le m\} \cup \{u_{10}u_{20}, u_{20}u_{30}, u_{30}u_{40}\}$. Then G has $4m+4$ vertices and $4m+3$ edges. Define $f : V(G) \to \{0, 1, 2, ..., D_{4m+3}\}$ as follows.

$f(u_{10}) = 0$

$f(u_{i0}) = f(u_{(i-1)0}) + 3i(i-3) + 7$ where $i = 2,3,4$

$f(u_{ij}) = f(u_{i0}) + 3[m^2(i^2 - 10i + 25) + j^2 + 2ijm + 5m(7-2j) - 7(im + j)] + 37$,where $i = 1,2,3,4$ and $1 \le j \le m$.

We shall prove that G admits centered hexagonal graceful labeling. From the definition, it is clear that $\max_{v\in V(G)} f(v)$ is D_{4m+3} and also $f(v)\in\{0,1,2,...,D_{4m+3}\}$. Also from the definition, all the vertices of G have different labeling. Hence f is one to one. It remains to show that the edge values are of the form $\{D_1, D_2, ..., D_{4m+3}\}$. The induced edge function $f^*: E(G)\rightarrow\{D_1, D_2, ..., D_{4m+3}\}$ is defined as follows.

$f^*(u_{10}u_{(i+1)0}) = 3i^2 - 3i + 1$, where $i = 1,2,3$

$f^*(u_{10}u_{1j}) = 3[8m(2m-j) + j^2 + 7(4m-j)] + 37$, where $1\le j\le m$

$f^*(u_{20}u_{2j}) = 3[3m(3m-2j) + j^2 + 7(3m-j)] + 37$, where $1\le j\le m$

$f^*(u_{30}u_{3j}) = 3[4m(m-j) + j^2 + 7(2m-j)] + 37$, where $1\le j\le m$

$f^*(u_{40}u_{4j}) = 3[m(m-2j) + j^2 + 7(m-j)] + 37$, where $1\le j\le m$

Clearly f^* is a bijection and $f^*(E(G)) = \{D_1, D_2, ..., D_{4m+2}, D_{4m+3}\}$. Therefore G admits centered hexagonal graceful labeling. Hence the uniform caterpillar $Cat_{4,m}$ is a centered hexagonal graceful graph.

Example 3.8: The centered hexagonal graceful labeling of the uniform caterpillar $Cat_{4,m}$ is given in figure 4.

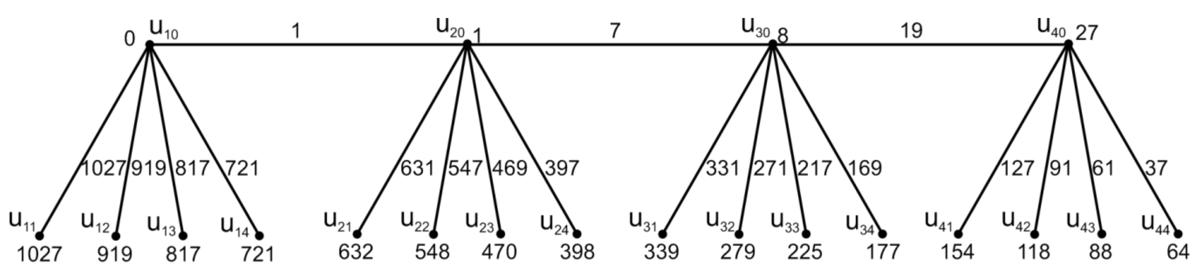

Figure 4

Theorem 3.9: The uniform caterpillar $Cat_{n,m} \forall n \geq 2$ is a centered hexagonal graceful graph.

Proof: Let $G=(V,E)$ be the uniform caterpillar $Cat_{n,m} \forall n \geq 2$ with the vertex set $V=\{u_{ij} : 1 \leq i \leq n;\ 0 \leq j \leq m\}$ and the edge set $E=\{u_{i0}\, u_{ij}\ : 1 \leq i \leq n; 1 \leq j \leq m\} \cup \{u_{i0} u_{(i+1)0} : 1 \leq i \leq n-1\}$. Then G has $nm+n$ vertices and $nm+n-1$ edges. Define $f : V(G) \rightarrow \{0,1,2,..., D_{nm+n-1}\}$ as follows.

$f(u_{10}) = 0$

$f(u_{i0}) = f(u_{(i-1)0}) + 3i(i-3) + 7$, where $2 \leq i \leq n$

$f(u_{ij}) = f(u_{i0}) + 3(nm+n+m-im-j)(nm+n+m-im-j-1)+1$, where $1 \leq i \leq n$ and $1 \leq j \leq m$.

We shall prove that G admits hexagonal graceful labeling. From the definition, it is clear that $\max_{v \in V(G)} f(v)$ is D_{nm+n-1} and also $f(v) \in \{0,1,2,..., D_{nm+n-1}\}$. Also from the definition, all the vertices of G have different labeling. Hence f is one to one. It remains to show that the edge values are of the form $\{D_1, D_2, ..., D_{nm+n-1}\}$. The induced edge function $f^* : E(G) \rightarrow \{D_1, D_2, ..., D_{nm+1}, D_{nm+2}, ..., D_{nm+n-1}\}$ is defined as follows.

$f^*(u_{i0}\, u_{(i+1)0}) = 3i(i-1)+1$, where $1 \leq i \leq n-1$

$f^*(u_{i0}\, u_{ij}) = 3(nm+n+m-im-j)(nm+n+m-im-j-1)+1$, where $1 \leq i \leq n$ *and* $1 \leq j \leq m$

Clearly f^* is a bijection $f^*(E(G)) = \{D_1, D_2, ..., D_{nm+1}, D_{nm+2}, ..., D_{nm+n-1}\}$. Therefore G admits centered hexagonal graceful labeling. Hence the uniform caterpillar $Cat_{n,m} \forall n \geq 2$ is a centered hexagonal graceful graph.

Example 3.10: The centered hexagonal graceful labeling of the uniform caterpillar $Cat_{5,3}$ is given in figure 5.

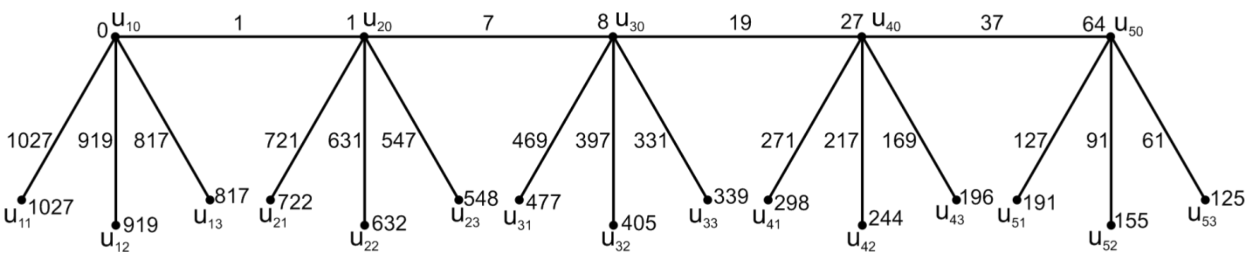

FIGURE 5

Conclusion:

In this paper, the centered hexagonal graceful labeling of caterpillar and uniform caterpillar graphs are studied. This work contributes several new results to the theory of graph labeling**.**

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